THE CONTENT OF CONTENT SEMANTICS

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ABSTRACT. In this paper, we investigate Brady's *content semantics*, which was an early attempt to get around the infamous incompleteness of the standard axiom systems for quantified relevant logics with regard to constant domain expansions of ternary relation semantics. We investigate this semantic framework by showing equivalence to a variation on a recently proposed algebraic version of semantics due to Mares and Goldblatt.

1. INTRODUCTION

As emphasized in the historical account prefacing [16], relevant logics often lacked a semantics. For years the propositional relevant logics yearned for a semantic interpretation. The ternary relational semantics by Sylvan (né Routley) and Meyer [16], and the algebraic semantics of Dunn [4] opened the floodgates to developing semantics generally for relevant logics. The ternary relational semantics, with the star developed from Sylvan and Plumwood (née Routley/Morell) [17], was the key player for relevant logicians. It was hoped this semantics could be easily generalized into semantics for first-order logics. This hope was dealt a direct blow by Fine [8], wherein it is written that the ternary relational semantics appended with a constant domain and the usual Tarskian truth conditions for the quantifiers validates too many formulas. That is, the logic **RQ** (and several related logics) presented as Hilbert style axiom systems is incomplete with respect to the most straightforward constant domain generalization of the ternary relational

We'd like to thank the editor for organising this special issue, and a reviewer for helpful suggestions. Our main thanks is to Ross Brady for decades of inspiring work in relevant and related logics. Tedder was financially supported by the Deutsche Forschungsgemeinschaft project TE 1611/1-1. Ferenz recognizes funding from RVO 67985807 and by the Czech Science Foundation project GA22-01137S.

semantics. Moreover, to this day no one knows whether or not this routine generalization of the ternary relational semantics is axiomatizable.[1]

While the Mares-Goldbatt semantics and generalizations are yet to be developed in the 21st century, two semantic approaches were given for which first-order relevant logics were complete. The first is a seemingly unatural (or at least complicated) yet genius semantics given by Fine [7]. The second, the focus of this paper, is the content semantics developed by Brady [1, 2] (and presented in [3]). Brady's content semantics extends the algebraic semantic approach. In essence the content semantics gives an admissible set of functions from a set into a algebraic structure. The presentation of the content semantics is quite dense and complicated, involving many notational variants of essentially the same thing. The major goal of this paper is to show that Brady's content semantics are fundamentally similar to MG Matrices (introduced below). A consequence of this (and the definition of the MG Matrices) is that the MG Matrices are a simpler, more natural presentation of essentially the same thing.

In what follows, we first complete the introductory material by giving some of the historical remarks surrounding the relevant semantic systems. In Section § 3 and § 4 we introduce MG Matrices and Brady's Content Semantics, respectively. Then, in § 5 we show that we can construct an MG Matrix from a given structure of content semantics, and the reverse direction is shown in § 6 (given that the MG matrix satisfies a certain *expressivity* constraint).

Brady's content semantics appears to be in part motivated by a rejection of fusion and the intensional/Ackkermann truth constant used in algebraic semantics. These purely algebraic operators only subtract from the naturalness of a semantics.

> A content semantics is essentially an algebraic-style semantics without those algebraic operators which take the semantics towards more or less conventional algebraic theories and away from being a more or less direct semantics of the logical concepts. [1, p. 111].

Brady distinguished his content semantics from other kinds of semantics. Content semantics differs from the Meyer, Dunn, Leblanc [15] semantics because in their structures any defined algebra that extends the propositional structure is "a *trivial* one in that all interpretations I satisfy the trivial condition, $I(A) \in T$, for all sentential instances A of quantificational axioms, where T is the truth filter" [1, p. 114]. Furthermore, the content semantics is not a generalization of the usual algebraic semantics for first-order logics.

> [T]he type of semantics presented here is different from the three types of classical algebraic semantics that have been used, viz the Rasiowa and Sikorski semantics using unrestricted generalized meets and joins..., the Halmos semantics using polyadic algebras..., and the Henkin-Monk-Tarski semantics using cylindric algebras... [1, p. 114]

[1]Though, it is the case that we know some things: its set of validities is closed under γ, as shown by Kripke [18] using semantic tableaux.

In particular, predicates aren't modeled as truth functions but as content functions: that is, functions from sequences of objects to contents.

> The main Difference in our semantics is that n-place functions from D^n to C are used to interpret n-place predicates, where D is the domain of individuals and C is the set of contents. [1, p. 114]

The use of content functions in Brady's semantics, as we will see below, requires keeping track of each occurrence of a free variable in a formula. We show that the structure of content functions can be reformulated equivalently as ω-ary functions similar to propositional functions.[2] This links the semantics to Halmos' polyadic algebra [10] and the Mares-Goldblatt semantics [13], the latter of which the MG Matrices below are motivated.

2. Language and Formulas

The language *Fm* is defined inductively as usual. Atomic formulas are defined out of a set of *terms Term*, which is the union of:

- A denumerable set of *variable letters Var*, denoted by x, perhaps with subscripts.
- A set of *name constants Con*.

and *predicate letters Pred*, each of which is denoted by P (perhaps with decorations), each of which is assigned a natural number as *arity*, made explicit as a superscript where necessary. Atomic formulas are of the form $P^n(\tau_1, \ldots, \tau_n)$ – where P has arity 0, it is a *propositional constant*. The set of predicate letters of arity n is denoted $Pred_n$.

Out of atomic formulas are constructed complex formulas by applications of the *connectives* $\neg, \wedge, \vee, \to$ (of arities 1,2,2,2, respectively) and *quantifiers* $\forall, \exists$ in the usual way. A formula A is said to have a variable *x free* when some subformula of A includes an occurrence of x which is not in the scope of a quantifier.

$\mathcal{A}, \mathcal{B}, \ldots$ will serve as metavariables over formulas.

3. MG Matrices

3.1. **Algebras.**

The MG Matrices are a matrix generalization of MG algebras introduced in [20]. These aim to capture an algebraic core found in the Mares-Goldblatt interpretation of the quantifiers in relational frames. This Mares-Goldblatt interpretation was introduced for the quantified relevant logics **RQ** and **QR** in [13], and

[2] This is required given how Brady defined his models. He could have defined them to employ ω-ary propositional functions taking variable assignments. Instead, however, Brady tied his various assignment and interpretation functions quite closely to the syntax, requiring us to bidge the gap. This bridge-building is the cause of much of the work we need to do here.

for quantified modal classical logic in [14].[3] The MG Matrices (and algebras) capture the essence of the Mares-Goldblatt interpretation of the quantifiers: a quantified formula $\forall x\mathcal{A}$ is interpreted as an (admissible) element which (i) entails each of its instances, and (ii) is entailed by anything satisfying (i).

Definition 3.1 (BB Matrix)**.** A *BB matrix* is a tuple $\langle\langle\mathcal{A},\neg,\wedge,\rightarrow\rangle,\mathcal{D}\rangle$ with $\mathcal{D}\subseteq\mathcal{A}$ such that:

(1) $\langle\mathcal{A},\neg,\wedge,\vee\rangle$ is a DeMorgan lattice, where $a\vee b=\neg(\neg a\wedge\neg b)$, with a lattice order $\leq$ defined as usual (i.e., $a\leq b$ iff $a\vee b=b$ iff $a\wedge b=a$)
(2) if $a\leq b$ then $c\rightarrow a\leq c\rightarrow b$
(3) if $a\leq b$ then $b\rightarrow c\leq a\rightarrow c$
(4) $a\leq b$ iff $\exists x\in\mathcal{D}\ (x\leq a\rightarrow b)$

Definition 3.2 (BB Algebra with $\mathtt{t}$)**.** A *BB algebra with t* is a tuple $\langle\mathcal{A},\neg,\wedge,\rightarrow,\mathtt{t}\rangle$ satisfying (1)–(3) of definition 3.1 and the following:

(4′) $a\leq b$ iff $\mathtt{t}\leq a\rightarrow b$

Combining the previous definitions, we obtain BB Matrices based on BB Algebras with $\mathtt{t}$.

Definition 3.3 (BB Matrix with $\mathtt{t}$)**.** A *BB Matrix with t* is a tuple $\langle\mathcal{A},\mathcal{D}\rangle$ where $\mathcal{A}$ is a BB Algebra with $\mathtt{t}$, $\mathcal{D}\subseteq\mathcal{A}$ is such that:

$$[\mathtt{t})=\{a\in\mathcal{A}\mid\mathtt{t}\leq a\}\subseteq\mathcal{D}$$

For the main results of the paper, we only use *BB* Matrices, though it should be noted certain things are simplifed by having $\mathtt{t}$ around.[4]

The logic **BB** can be given a Hilbert-style axiomatization as in [12, 11], wherein we find a strong motivation for the logic. It is fairly straightforward to show that **BB** is sound and complete w.r.t. the class of BB matrices, fixing (propositional) interpretations so that a formula is satisfied on a matrix-interpretation pair just in case the value of the formula belongs to $\mathcal{D}$.

The following definitions instantiate a matricial variant of the MG Structures of [20].

Definition 3.4 (MG Matrix)**.** An *MG Matrix* is a tuple $\mathfrak{A}=\langle A^{\mathfrak{N}},A^{\mathfrak{C}},D,PF,h\rangle$ where:

(1) $A^{\mathfrak{N}}$ is a BB Matrix. (the *seam*)
(2) $A^{\mathfrak{C}}$ is a complete lattice-ordered matrix of the same type as $A^{\mathfrak{N}}$.[5] (the *nugget*)
(3) $D\neq 0$.
(4) $PF\subseteq A^{\mathfrak{N}(D^{\omega})}$ such that:
  (a) For any $\varphi\in PF$, there is a $\neg\varphi\in PF$, where for every $f\in D^{\omega}$ $\neg\varphi f=\neg^{\mathfrak{N}}(\varphi f)$

[3] It has been found a powerful tool for first-order relevant logics, and was extended to a wide range of logics for relational frames [6], neighbourhood frames [21], propositionally quantified relevant logics [9], and identity [19, 5].

[4] For instance, with this we don't need constraing (5)(d) in Def. 3.4 below.

[5] Superscripted $\mathfrak{C},\mathfrak{N}$ distinguish in which lattice meets and joins are evaluated.

(b) For any $\varphi, \psi \in PF$, there is $\otimes(\varphi, \psi) \in PF$, for $\otimes \in \{\rightarrow, \wedge\}$ where, for any $f \in D^\omega$, $(\varphi \otimes \psi)f = \varphi f \otimes^{\mathfrak{N}} \psi f$

(c) For every $n \in \omega$, there is $\forall_n, \exists_n \in PF$ (note the requirements below).

(5) $h : A^{\mathfrak{N}} \longrightarrow A^{\mathfrak{C}}$ is a homomorphism where:

(a) $a \leq^{\mathfrak{N}} b \Leftrightarrow h(a) \leq^{\mathfrak{C}} h(b)$

(b) $\mathcal{D}^{\mathfrak{N}} = h^{-1}\mathcal{D}^{\mathfrak{C}}$ (i.e., h is a matrix homomorphism)[6]

(c) The following identities hold (where $\otimes \in \{\rightarrow, \wedge, \vee\}$):

$$h(\neg^{\mathfrak{N}}(a)) = \neg^{\mathfrak{C}} h(a)$$

$$h(a \otimes^{\mathfrak{N}} b) = h(a) \otimes^{\mathfrak{C}} h(b)$$

$$h((\forall_n \varphi)f) = \bigvee^{\mathfrak{C}} \{a \in ran(h) : a \leq^{\mathfrak{C}} \bigwedge^{\mathfrak{C}}_{f' \sim_n f} h(\varphi f')\}$$

$$h((\exists_n \varphi)f) = \bigwedge^{\mathfrak{C}} \{a \in ran(h) : \bigvee^{\mathfrak{C}}_{f' \sim_n f} h(\varphi f') \leq^{\mathfrak{C}} a\}$$

(d) If $\{h(\varphi f') \mid f' \sim_n f\} \subseteq \mathcal{D}^{\mathfrak{N}}$ then $(\forall_n \varphi)f \in \mathcal{D}^{\mathfrak{N}}$.[7]

Let's unpack this definition a bit, to see how the above structures can be understood as, as we've claimed they are, a matricial variant of the MG frames to be found in [13].[8] The key move for our purposes here is that in an MG frame, we have both a set of points W (along with its powerset $wp(W)$) as well as a set of *admissible propositions* $Prop$ where we may have that $Prop \subsetneq \mathscr{P}(W)$. Elements of $Prop$ serve as interpretants of closed formulas, and open formulas are interpreted by functions taking variable assignments D^ω into $Prop$ – the set of such functions is designated $PropFun$, and is also allowed to omit some functions in $Prop^{D^\omega}$, as long as there are enough to interpret all open formulas. The key move here is that the divergence between $Prop$ and $\mathscr{P}(W)$ allows one to interpret the quantifiers as something other than simple intersections and unions, which move leads to Fine's [8] incompleteness result. Instead, we can define an operation to interpret formulas like $\forall x \mathcal{A}$ whose value is computed *in terms of* the intersection of the values of $\mathcal{A}$ (which inhabits $\mathscr{P}(W)$, but need not inhabit $Prop$), but which must inhabit $Prop$. That is, we can see $\mathscr{P}(W)$ as a *completion* of $Prop$ (as a lattice), and interpret quantified expressions in the latter while appealing to the structure of $\mathscr{P}(W)$ to pin down the correct interpretation.

MG Matrices capture this move by introducing a set $A^{\mathfrak{N}}$ which stands in for $Prop$ and another, $A^{\mathfrak{C}}$, which stands in for $\mathscr{P}(W)$. We require that $A^{\mathfrak{C}}$ be a completion (by the matrix homomorphism h) of $A^{\mathfrak{N}}$ which preserves the propositional operators,

[6]This is the distinctive feature of MG *matrices*, and no such condition is required in the more general algebraic setting.

[7]This condition is needed to ensure that the set of formulas satisfied in a model on an MG matrix is closed under the rule of *universal generalization*, i.e. that if $\mathcal{A}$ is valid on the algebra, then so is $\forall x \mathcal{A}$.

[8]For further details of how the MG Matrices/Algebras are related to the Mares-Goldblatt interpretation the reader is directed to [20].

and which satisfies two further equations w.r.t. h – namely, the last two displayed equations in condition (5.c). These equations directly express the interpretation of quantifiers introduced in [13], but in this more general matrix setting. The other conditions are there to ensure that we can interpret formulas in $A^{\mathfrak{N}}$ (with PF being our intermediary, as in *PropFun* in MG frames). The superscripts on the order, meets, and joins, are there to indicate that they are to be interpreted as in $A^{\mathfrak{C}}$, where all the desired meets and joins are guaranteed to live.

With this background in mind, let us fix the definition of models.

Definition 3.5 (MG Model)**.** An *MG Model* is a tuple $\mathfrak{M} = \langle \mathfrak{A}, M \rangle$ where M is a multitype function, of types $Con \longrightarrow D$ and $D^n \longrightarrow A^{\mathfrak{N}}$, and for any $f \in D^\omega$ we define $M_f : Term \longrightarrow D$ by $M(c) = c$ for constants and $M_f(x_n) = fn$ for variables. Moreover, M is used to construct a homomorphism $[\![-]\!]^{\mathfrak{M}} : Fm \longrightarrow (A^{\mathfrak{N}})^{(D^\omega)}$ such that:[9]

$$\begin{aligned} [\![P(\tau_1, \dots, \tau_n)]\!]^{\mathfrak{M}}_f &= M(P)(M_f(\tau_1), \dots, M_f(\tau_n)) \\ [\![\otimes(\mathcal{A}_1, \dots, \mathcal{A}_n)]\!]^{\mathfrak{M}}_f &= \otimes([\![\mathcal{A}_1,]\!]^{\mathfrak{M}}_f, \dots, [\![\mathcal{A}_n,]\!]^{\mathfrak{M}}_f) \quad \text{for } \otimes \in \{\neg, \wedge, \vee, \rightarrow\} \\ [\![\forall x_n \mathcal{A}]\!]^{\mathfrak{M}}_f &= (\forall_n [\![\mathcal{A}]\!]^{\mathfrak{M}}) f \\ [\![\exists x_n \mathcal{A}]\!]^{\mathfrak{M}}_f &= (\exists_n [\![\mathcal{A}]\!]^{\mathfrak{M}}) f \end{aligned}$$

For a model $\langle \mathfrak{A}, M \rangle$ (with $\mathfrak{M}$ constructed from M), we define $\vDash^{\mathfrak{A},\mathfrak{M}}_f \mathcal{A}$ to denote $[\![\mathcal{A}]\!]^{\mathfrak{M}}_f \in \mathcal{D}$. We say that $\vDash^{\mathfrak{A},\mathfrak{M}} \mathcal{A}$ iff $\vDash^{\mathfrak{A},\mathfrak{M}}_f \mathcal{A}$ for each $f \in D^\omega$. Finally, $\vDash^{\mathfrak{A}} \mathcal{A}$ just in case $\vDash^{\mathfrak{A},\mathfrak{M}} \mathcal{A}$ for each $\mathfrak{M}$ based on $\mathfrak{A}$.

4. Content Semantics

Brady's presentation of the content semantics is often over-notated: he defines, in essence, several notational variants and setting conditions with distinct variants. The most significant class of these variants correspond to the (admissible) *content functions* he introduces.[10] Much of our effort is in reducing content functions to propositional functions, to allow for simple comparison. A *content function*, more specifically, is a function from ordered n-tuples of individuals into the set of *contents*. In particular, for any positive integer n we have a set of functions from D^n into C, for the domain of individuals D and the contents C. We denote the set of such n-ary functions in a model structure $\mathbb{F}^n$. For any given model structure, we define $\mathbb{F} = \bigcup\{\mathbb{F}^n \mid n \in \mathbb{N}\}$. Note that $\mathbb{F}^0$ is a set of constant functions, so for convenience we will just assume that $\mathbb{F}^0 \subseteq C$.

[9]We write $[\![-]\!]^{\mathfrak{M}}_f$ instead of $[\![-]\!]^{\mathfrak{M}}(f)$, where convenient.

[10]The term *content functions* is our own, chosen to contrast with *propositional function*. Note also: the set of functions is not guaranteed to be *full* — that is, it needn't contain every such function — and so the content semantics includes the power of general frames. This is a key element in the difference between the most straightforward constant domain extension of ternary relational semantics, see, e.g. [8], and ones with the Mares-Goldblatt interpretation, where fullness is also not assumed.

The set $\mathbb{F}$ is required to be closed under many operations. For negation, we have a unary function $*$ on C, from which we define the following: where $F^n \in \mathbb{F}^n$, $F^{n*}(a_1, \ldots, a_n) =_{df} F^n(a_1, \ldots, a_n)^*$. For convenience, we will write $\vec{a}_n$ to represent a list of a_i's of length n. For any binary $\otimes \in \{\sqcup, \sqcap, \Rightarrow\}$, for n-place F^n and m-place function G^m, we define the $n+m$-ary function $(F^n \otimes G^m)$ by $(F^n \otimes G^m)(\vec{a}_n, \vec{b}_m) =_{df} F^n(\vec{a}_n) \otimes G^m(\vec{b}_m)$. Note that each occurrence of a free variable in a formula, or rather a function corresponding to a (set of) variables, is treated separately.

For the quantifiers, we need to introduce a little more machinery. Let $\langle \vec{j} \rangle_m$ denote an ordered set of positive integers with length m. Given $m \leq n$, $\vec{a}_{n-m}$, and $b \in D$, we define ${}^{b/\langle \vec{j} \rangle_m}(\vec{a}_{n-m})$ as the sequence of elements of D consisting of b in each of the $\langle \vec{j} \rangle_m$ positions, and elements of $\vec{a}_{n-m}$ in the remaining positions, keeping the order. For example, if $\langle \vec{j} \rangle_2 = \{1, 4\}$ and $a_1 = a_2 = c$, then

$$ {}^{b/\langle \vec{j} \rangle_2}(a_1, a_2) = \langle b, c, c, b \rangle $$

i.e., substituting b into the first and fourth position. Similarly, we define an $n-m$-place function $F^{n\ b/\langle \vec{j} \rangle_m}$ by

$$ F^{n\ b/\langle \vec{j} \rangle_m}(\vec{a}_{n-m}) = F^n[{}^{b/\langle \vec{j} \rangle_m}(\vec{a}_{n-m})] $$

for all $a_1, \ldots, a_n \in D$.

We define a set V to be a set of sets of 'instances' of functions applied to lists of individuals. That is:

$$ V =_{df} \{\{F^{n\ b/\langle \vec{j} \rangle_m}(\vec{a}_{n-m}) \mid b \in D\} \mid n, m \in \mathbb{Z}^+, m \leq n, F^n \in \mathbb{F}^n, \ \&\ a_1, \ldots, a_{n-m} \in D\} $$

The generalized meets and joins interpreting the quantifiers have the domain V. That is, e.g., the generalized meet of all instances of F^n for each b substitutes for a subset j of 'free variables'.[11] As will be made clear, we will require C to be closed under certain generalized meets. Namely, we require that $\bigsqcap\{F^{n\ b/\langle \vec{j} \rangle_m}(\vec{a}_{n-m}) \mid b \in D\} \in C$. We can now define the $n-m$-place function corresponding to the generalized meet operation. Such a function, written $\bigsqcap\{F^{n\ b/\langle \vec{j} \rangle_m} \mid b \in D\}$ is defined by

$$ \bigsqcap\{F^{n\ b/\langle \vec{j} \rangle_m} \mid b \in D\}(\vec{a}_{n-m}) =_{df} \bigsqcap\{F^{n\ b/\langle \vec{j} \rangle_m}(\vec{a}_{n-m}) \mid b \in D\} $$

for all $a_1, \ldots, a_{n-m} \in D$.

The final cluster of concepts we need surround that of a variable assignment, namely a ω-length list of elements of D. A variable assignment $s : \mathbb{Z}^+ \longrightarrow D$ is a variable assignment, the set of which Brady denotes by S, and which we will denote as D^ω. We require that if $s \in D^\omega$ and $F^n \in \mathbb{F}^n$, then $F^n(s(i_1), \ldots s(i_n)) \in C$, where $i_i \in \mathbb{Z}^+$.

For any variable assignment s, the k-variant of s assigning b to the kth position, denoted by $s^{b/k}$ is the same as s but such that $s^{b/k}(k) = b$. As above, we use the relation $\sim_k$ to denote the relation of being k-variant.

[11] Note here that each a_i in a given instance is an element of D, and that the jth place in ${}^{b/\langle \vec{j} \rangle_m}(\vec{a}_{n-m})$ was not 'filled in' with either a variable or constant 'before' the 'substitution'. Each list ${}^{b/\langle \vec{j} \rangle_m}(\vec{a}_{n-m})$ was created from scratch. Rather, the terminology 'free variable' anticipates a role in modeling formulas in the interpretations to follow.

Given a function $F^n \in \mathbb{F}^n$ and $s \in D^\omega$, we can define an n-ary function from positive integers to an element of C, which acts as a sort of auxiliary function with no explicit mention of D, as follows:[12]

$$F^n s(\vec{i}_n) = F^n s(i_1, \ldots, i_n) = F^n(s(i_1), \ldots, s(i_n)), \text{ for all } i_1, \ldots, i_n \in \mathbb{Z}^+$$

Corresponding to variables that are not free (or do not occur) is the notion of k-constancy or independence. An expression $F^n s(\vec{i}_n)$ is *k-constant* if k does not occur in $(\vec{i}_n)$, for $k \in \mathbb{Z}^+$. If an expression $F^n s(\vec{i}_n)$ is k-constant, then $F^n s(\vec{i}_n) = F^n s^{b/k}(\vec{i}_n)$.

Remark 4.1. Note the difference between the sequence of integers $\vec{i}_n$, in which the same integer can appear more than once, and the set of integers $\langle \vec{i} \rangle_n$, which contains n different integers without an ordering.

We can now define a model structure.

Definition 4.2 (Unreduced **L**-model structure)**.** A(n *unreduced)* ***L****-model structure* (*BBQ m.s., content structure*) is a tuple

$$\mathfrak{S} = \langle C, T, D, \mathbb{F}, \leq, *, \sqcap, \sqcup, \Rightarrow, \bigsqcap, \bigsqcup \rangle$$

where

(1) C is a non-empty set of contents
(2) $T \subseteq C$ is the set of true contents
(3) D is a non-empty set (the domain)
(4) $\leq \subseteq C^2$
(5) $\sqcap, \sqcup$, and $\Rightarrow$ are functions of type $C^2 \longrightarrow C$
(6) $* : C \longrightarrow C$
(7) $\bigsqcup$ and $\bigsqcap$ are functions of type $V \longrightarrow C$
(8) $\mathbb{F}$ is defined as above.

The conditions an unreduced **L**-model structure satisfies are divided into closure conditions and semantic postulates.

Closure conditions (on $\mathbb{F}^n$):

(c1) If $F^n \in \mathbb{F}^n$, then $\mathbb{F}^{n^*} \in \mathbb{F}^n$.
(c2) If $F^n \in \mathbb{F}^n$ and $G^n \in \mathbb{F}^m$, then $F^n \sqcup G^m,\ F^n \sqcap G^m,\ F^n \Rightarrow G^m \in \mathbb{F}^{n+m}$.
(c3) If $F^n \in \mathbb{F}^n$ and $\langle \vec{j} \rangle_m \subseteq \{1, \ldots, n\}$, then $\bigsqcap\{F^{n\ b/\langle \vec{j} \rangle_m} \mid b \in D\} \in \mathbb{F}^{n-m}$ and $\bigsqcup\{F^{n\ b/\langle \vec{j} \rangle_m} \mid b \in D\} \in \mathbb{F}^{n-m}$.

Semantic postulates:

For $c, d, e \in C$, $F^n \in \mathbb{F}^n$, $G^m \in \mathbb{F}^m$, $\vec{i}_n \in (\mathbb{Z}^+)^n$, $\vec{j}_m \in (\mathbb{Z}^+)^m$, $k \in \mathbb{Z}^+$:

(p1) $\leq$ is a partial ordering on C.
(p2) (a) $c \sqcap d \leq c$; $c \sqcap d \leq d$.
(b) If $c \leq d$ and $c \leq e$, then $c \leq d \sqcap e$.
(p3) (a) $c \leq c \sqcup d$; $d \leq c \sqcup d$.
(b) If $c \leq e$ and $d \leq e$, then $c \sqcup d \leq e$.
(p4) $c \sqcap (d \sqcup e) \leq (c \sqcap d) \sqcup (c \sqcap e)$.
(p5) (a) $c^{**} = c$.

[12] As per the definition, if $n = 0$, $F^0 s = F^0 (\in C)$.

(b) If $c \leq d$ then $d^* \leq c^*$.
(p6) T is closed under $\sqcap$ and is an $\leq$-upset.
(p7) (a) $c \leq d$ iff $c \Rightarrow d \in T$.
(b) If $c \leq d$ then $d \Rightarrow e \leq c \Rightarrow e$.
(c) If $c \leq d$ then $e \Rightarrow c \leq e \Rightarrow d$.
(p8) (a) $\bigsqcap\{F^n s^{b/k}(\vec{i}_n) \mid b \in D\} \leq F^n s^{b/k}(\vec{i}_n)$, for all $b \in D$.
(b) If $F^n s(\vec{i}_n)$ is k-constant and $F^n s(\vec{i}_n) \leq G^m s^{b/k}(\vec{j}_m)$ for each $b \in D$, then $F^n s(\vec{i}_n) \leq \bigsqcap\{G^m s^{b/k}(\vec{j}_m) \mid b \in D\}$
(p9) (a) $F^n s^{b/k}(\vec{i}_n) \leq \bigsqcup\{F^n s^{b/k}(\vec{i}_n) \mid b \in D\}$, for all $b \in D$.
(b) If $G^m s(\vec{j}_m)$ is k-constant and $F^n s^{b/k}(\vec{i}_n) \leq G^m s(\vec{j}_m)$ for each $b \in D$, then $\bigsqcup\{F^n s^{b/k}(\vec{i}_n) \mid b \in D\} \leq G^m s(\vec{j}_m)$
(p10) $\bigsqcap\{(F^n \sqcup G^m)s^{b/k}(\vec{i}_n, \vec{j}_m) \mid b \in D\} \leq F^n s(\vec{i}_n) \sqcup \bigsqcap\{G^m s^{b/k}(\vec{j}_m) \mid b \in D\})$, where $F^n s(\vec{i}_n)$ is k-constant.
(p11) If $F^n s^{b/k}(\vec{i}_n) \in T$, for all $b \in D$, then $\bigsqcap\{F^n s^{b/k}(\vec{i}_n) \mid b \in D\} \in T$

Definition 4.3 (Interpretations)**.** An *interpretation* I on a BBQ m.s. $\mathfrak{S}$ is a multi-type function on predicates and variables defined as follows:
(I1) $I(P^n) \in \mathbb{F}^n$, for $P^n \in Pred_n$.
(I2) $I(x_k), I(c) \in D$.[13]
and this function is defined inductively on formulas as follows:
(I3) $I(P^n(x_{i_1}, \ldots, x_{i_1})) = I(P^n)(I(x_{i_1}), \ldots, I(x_{i_1}))$.
(I4) $I(\neg\mathcal{A}) = (I\mathcal{A})^*$.
(I5) $I(\mathcal{A} \wedge \mathcal{B}) = I(\mathcal{A}) \sqcap I(\mathcal{B})$.
(I6) $I(\mathcal{A} \rightarrow \mathcal{B}) = I(\mathcal{A}) \Rightarrow I(\mathcal{B})$.
(I7) $I(\forall x_k \mathcal{A}) = \bigsqcap\{I^{b/x_k}(\mathcal{A}) : b \in D\}$, where I^{b/x_k} is the x_k-variant of I assigning b to x_k.[14]

Brady shows that $I^{b/x_k}(\mathcal{A})$ has the form $F^{n\ b/\langle\vec{j}\rangle_m}(\vec{a}_{n-m})$, and thus $\bigsqcap\{I^{b/x_k}(\mathcal{A}) : b \in D\}$ is well-defined.

We say that a formula $\mathcal{A}$ is *true on the interpretation I on BBQ m.s.* $\mathfrak{S}$ when $I(\mathcal{A}) \in T$; *valid on* $\mathfrak{S}$ when true on every interpretation on $\mathfrak{S}$; *valid* (simpliciter) when it is valid on every BBQ m.s. $\mathfrak{S}$. Brady shows soundness and completeness w.r.t. the logic **BBQ**.

5. Bunching Brady into the Matrix

We begin by showing how to transform models build on Brady's structures into models built on MG matrices, preserving satisfaction. The key theorem is:

Theorem 5.1. For any unreduced L-model structure $\mathfrak{S}$, there is a MG Matrix $\mathfrak{A}$ such that for any interpretation I on $\mathfrak{S}$, there is a corresponding evaluation $[\![-]\!] : Fm \longrightarrow PF$ such that $[\![\mathcal{A}]\!] \in \mathcal{D}$ iff $I(\mathcal{A}) \in T$.

The remainder of this section is dedicated to a proof of this theorem. The general route is by constructing an MG Matrix by taking the seam ($A^{\mathfrak{C}}$) to be a certain well-behaved *completion* of C (with the required operators), and taking the nugget ($A^{\mathfrak{N}}$) to be the set C itself. The

[13]Technically speaking, Brady does not include constants. However, out inclusion of constants here is easily handled by his **L**-model structures.
[14]A x_k variant differs only in the obvious case for ($I2$). Note also that Brady essentially relates the interpretation-varying semantics with satisfaction semantics.

first thing we need to do is abstract propositional functions from D^ω into propositions. In our target structures the set of propositions is just the set C from the content semantics. Brady's semantics does not contain functions of the correct type, but such functions can be obtained from Brady's machinery.

Brady defined auxiliary functions that combines a variable assignment s (i.e., $s \in D^\omega$) with a function F^n. The result is a function $F^n s$ that takes as argument a list of n positive integers. This list of positive integers represents variables: whereas the function F^n takes objects of D as input, $F^n s$ takes lists of positive integers. In order to obtain a propositional function, given any F^n, we flip the fixed input.

Definition 5.2 (PFs from Brady)**.** Suppose we have an unreduced L-model structure $\mathfrak{S}$ wherein $F^n \in \mathbb{F}$ and $\vec{i}_n$ is a n-length sequence of positive integers. We define a propositional function $F^n_{\vec{i}} : D^\omega \longrightarrow C$ by setting:

$$F^n_{\vec{i}_n} f = F^n f(\vec{i}_n) \qquad \text{for all } f \in D^\omega$$

The set of all such functions relative to $\mathfrak{S}$, i.e.:

$$PF(\mathfrak{S}) = \{F^n_{\vec{i}_n} \mid F^n \in \mathbb{F} \ \& \ \vec{i}_n \in (\mathbb{Z}^+)^n\}$$

we call the *set of propositional functions generated by* $\mathfrak{S}$.

For brevity, we often omit the natural number subscript n on the subscript $\vec{i}_n$, as it is (over)-determined by the function F^n to which it is attached.

We will define operations $\neg\vee, \wedge, \rightarrow$ on elements of $PF(\mathfrak{S})$, for a given $\mathfrak{S}$, by $\forall f \in D^\omega, \forall \varphi_1, \ldots, \varphi_n \in PF(\mathfrak{S})\big(\otimes(\varphi_1, \ldots, \varphi_n)f = \otimes(\varphi_1 f, \ldots, \varphi_n f)\big)$ (where we 'forget' the notational differences between $\sqcap$ and $\wedge$ and similarly for $\sqcup$ and $\vee$). This 'lifts' the propositional operators onto the resulting propositional functions. For the quantifiers, we will define an operator relative to the matrices we construct: that is, relative to the completion of C. For now, we record the following.

Lemma 5.3. For any unreduced L-model structure $\mathfrak{S}$, the set $PF(\mathfrak{S})$ is closed under $\neg, \wedge, \vee, \rightarrow$.

Proof. Suppose that $\varphi, \psi \in PF(\mathfrak{S})$. Then, by the definition, for any $f \in D^\omega$, we have that $\varphi f = F^n_{\vec{i}} f (= F^n f(\vec{i}))$ and $\psi f = G^m_{\vec{j}} (= G^m f(\vec{j}))$ for some $\vec{i} \in (\mathbb{Z}^+)^n$, $\vec{j} \in (\mathbb{Z}^+)^m$, and $F^n, G^m \in \mathbb{F}$. We want to show that $\varphi \wedge \psi, \varphi \vee \psi, \varphi \rightarrow \psi, \neg\varphi \in PF(\mathfrak{S})$. We give the case for only $\varphi \wedge \psi$: all other cases are similar.

We have that $F^n f(\vec{i}) \sqcap G^m f(\vec{j}) = F^n(f(\vec{i})) \sqcap G^m(f(\vec{i}))$ by the definition of functions $F^n f$ and $G^m f$. (That is, where we write $f(\vec{i})$ for $(f(i_1), \ldots, f(i_n))$.) From the conditions on $PF(\mathfrak{S})$ (specifically those listed in [1, p. 117]), we have that $F^n(f(\vec{i})) \sqcap G^m(f(\vec{i})) = F^n \sqcap G^m f(\vec{i}, \vec{j})$. Since $\vec{i}, \vec{j}$ is a $n+m$-length list of positive integers and $F^n \sqcap G^m \in \mathbb{F}$ by the closure conditions of content semantics, the result follows with $(\varphi \wedge \psi) = (F^n \wedge G^m)_{\vec{i},\vec{j}} \in PF(\mathfrak{S})$. ☐

For the quantifiers, we introduce a completion of the structure C for any $PF(\mathfrak{S})$.

Definition 5.4. Given the set of prime filters on C in a structure $\mathfrak{S}$ (denoted here as the set $C^\uparrow$), we define the set of $\subseteq$-upsets of these elements ($\hat{C} = \{X \subseteq C^\uparrow \mid a \in X \ \& \ a \subseteq b \text{ implies } b \in X\}$. For any element $a \in C$ we set $\hat{a} = \{X \in C^\uparrow \mid a \in X\}$. It is easy to check that $\hat{a}$ is an element of $\hat{C}$. We further lift the propositional operators $\otimes \in \{\sqcap, \sqcup, \Rightarrow, *\}$ by:

$\otimes(\hat{a}_1,\ldots,\hat{a}_n) = (\widehat{\otimes(\vec{a}_n)})$.

Consider a function h where $h(a) = \hat{a}$, for each $a \in C$. For each $n \in \omega$ and $\varphi \in PF(\mathfrak{S})$ and $f \in D^\omega$, we define:

$$(\forall_n \varphi)f = \bigvee^{\hat{C}} \{\hat{a} \in ran(h) \mid \hat{a} \subseteq \bigwedge^{\hat{C}}_{f' \sim_n f} h(\varphi f')\}$$

$$(\exists_n \varphi)f = \bigwedge^{\hat{C}} \{\hat{a} \in ran(h) \mid \bigvee^{\hat{C}}_{f' \sim_n f} h(\varphi f') \subseteq \hat{a}\}$$

The following lemma essentially says that, relative to the completion defined, the set $PF(\mathfrak{S})$ (dependent only on $\mathfrak{S}$ and not this particular completion) is closed under the quantifiers.

Lemma 5.5. Suppose that $\mathfrak{S}$ is an unreduced L-model structure and that we have completion $\hat{C}$ of C. Then the set $PF(\mathfrak{S})$ is closed under $\forall_k, \exists_k$ for each $k \in \omega$.

Proof. We show only the case for $\forall_k$. Suppose that $\varphi \in PF(\mathfrak{S})$ and $k \in \omega$. Then φ corresponds to some $F^n \in \mathbb{F}$; namely for any $f \in D^\omega$, $\varphi f = F^n_{\vec{i}} f = F^n(f(\vec{i}_n))$ where $(f(\vec{i})_n) \in D^n$. The reader is reminded that a variable assignment assigns to x_k the object it maps k to, as a variable assignment is a function from ω into D. Thus, we know which of $\vec{i}_n$ are quantified over (in a semantic way) by the quantifier $\forall_k$ in $\forall_k \varphi$.

Thus, consider the function $\prod\{F^{n-m\ b/\{j^k_1,\ldots,j^k_m\}} : b \in D\}_{i\vec{/}k}$, where $j^k_1,\ldots,j^k_m$ is the set of the positions of each occurrence of k in $\vec{i}$ and $i\vec{/}k$ is $\vec{i}$ with each of the m occurrences of k removed. By (c3) and the definition of $PF(\mathfrak{S})$, we have that $\prod\{F^{n-m\ b/\{j^k_1,\ldots,j^k_m\}} \mid b \in D\}_{i\vec{/}k} \in PF(\mathfrak{S})$. It remains to show (because h^{-1} is a function on the range of h) the following identity for each $f \in D^\omega$:

$$h[\prod\{F^{n-m\ b/\{j^k_1,\ldots,j^k_m\}} : b \in D\}_{i\vec{/}k} f] = \bigvee^{\hat{C}}\{\hat{a} \in ran(h) : \hat{a} \subseteq \bigwedge^{\hat{C}}_{f' \sim_k f} h(\varphi f')\}$$

We have the following sequence of equivalent elements:

$$\begin{aligned}
&\prod\{F^{n\ b/\{j^k_1,\ldots,j^k_m\}} : b \in D\}^{n-m}_{i\vec{/}k} f && \\
&= \prod\{F^{n\ b/\{j^k_1,\ldots,j^k_m\}} : b \in D\}^{n-m}_{f} i\vec{/}k && \text{DF 5.2}\\
&= \prod\{F^{n\ b/\{j^k_1,\ldots,j^k_m\}} : b \in D\}^{n-m}(f(i_1),\ldots,f(i_{n-m})) && \text{Notation}\\
&= \prod\{F^{n\ b/\{j^k_1,\ldots,j^k_m\}}(f^b/k(i_1),\ldots,f^b/k(i_{n-m})) : b \in D\} && \text{DF of } n-m \text{ function}\\
& && s = s^{b/k} \text{ on each } i\\
&= \prod\{F^{n\ b/\{j^k_1,\ldots,j^k_m\}}(s^{b/k}(i_1),\ldots,s^{b/k}(i_{n-m})) : b \in D\} && \text{DF of } n-m \text{ function}\\
&= \prod\{F^n f^{b/k}(\vec{i}) : b \in D\} && \text{Notation}
\end{aligned}$$

Thus, by conditions (p8).(a) on the content semantics and condition (5).(a) on h, and properties of the defined $\forall_n$, we have the left-to-right ($\leq$) direction of the target equality.

For the other direction: we can apply (p8).(b). Each instance of $\forall_k \varphi$ is implies by $\forall_k \varphi$. By the properties of h, we have then that $h^{-1}(\bigvee^{\hat{C}}\{\hat{a} \in ran(h) \mid \hat{a} \subseteq \bigwedge^{\hat{C}}_{f' \sim_k f} h(\varphi f')\}) \leq \varphi f'$, for each $f' \sim_k f$, and then applying (p8).(b) gives the desired inequation. □

We can now easily define the corresponding MG matrix for a structure of content semantics.

Definition 5.6 (Corresponding MG Matrix)**.** Let $\mathfrak{S} = \langle C, T, D, \mathbb{F}, \leq, *, \sqcap, \sqcup, \Rightarrow, \bigsqcap, \bigsqcup \rangle\rangle$ be an unreduced L-model structure. The corresponding MG Matrix $\mathfrak{A}^{\mathfrak{S}}$ is defined by:

(1) $A^{\mathfrak{N}} = \langle C, *, \sqcap, \sqcup, \Rightarrow, T\rangle$.
(2) $A^{\mathfrak{C}}$ is defined by:
 (a) $A^{\mathfrak{C}} = \hat{C}$,
 (b) the operations are $*, \sqcap, \sqcup$, and $\Rightarrow$ lifted to $\hat{C}$.
 (c) the operations $\forall_n$ and $\exists_n$ are defined as above.
(3) $D = D$.
(4) $PF = PF(\mathfrak{S})$.
(5) h is defined by $h(a) = \hat{a}$.

Lemma 5.7. A Corresponding MG Matrix is an MG Matrix.

Proof. Straightforwardly from the constraints on C, C is a set closed under $\neg, \wedge, \vee, \rightarrow$, (with a distributive DeMorgan lattice reduct, from property (p4) on content semantics, and so ordered by the usual definable $\leq$). Moreover, by condition (p6) the set T is an upset of designated values.

The usual arguments show that $\hat{C}$ is a complete lattice ordered by $\subseteq$. We trivially have that D is non-empty.

Lemmas 5.3 and 5.5 demonstrated that PF is closed under the propositional connectives, and that the required equalities hold. That is, conditions (1)–(5) of Def. 3.4 are satisfied.

Moreover, there are operators $\forall_n$ and $\exists_n$ in PF, for each $n \in \omega$. It remains to be shown that h is suitably defined. We have already seen that $h(a) = \hat{a}$ satisfies condition (5)(c) of Def. 3.4. This leaves conditions (a) and (b), i.e. that $a \leq^{\mathfrak{N}} b \iff \hat{a} \leq^{\mathfrak{C}} \hat{b}$ and that $\mathcal{T}^{\mathfrak{N}} = h^{-1}\mathcal{T}^{\mathfrak{C}}$. The former is immediate from the fact that $\hat{a} \sqcap \hat{b} = \widehat{a \sqcap b}$, and that the lattice order is defined in both algebras. For the latter, we fix $\mathcal{D}^{\mathfrak{C}} = \{\hat{a} \mid a \in T^{\mathfrak{N}}\}$. □

Lemma 5.8. For any BBQ m.s. $\mathfrak{S}$ there is a corresponding evaluation $[\![-]\!]^I : Fm \longrightarrow PF$ such that for every interpretation I (on $\mathfrak{S}$) and variable assignment f^I (where $f^I(x_n) = I(x_n)$) we have $[\![\mathcal{A}]\!]_{f^I} = I(\mathcal{A})$.

Proof. Given that Brady's interpretations bake in interpretations of the variables, each BBQ m.s. interpretation will fix a particular variable assignment f^I given the interpretation's valuation of the variables. That is,

$$f^I(x_k) =_{df} I(x_k)$$

For a given interpretation I, we also define the corresponding multi-type function M^I by setting $M^I(P^n) =_{df} I(P^n)$, and by the usual definition of M^I_f applied to a variable. (Reminder: there are no constants in Brady's semantics.) From here, we construct the assignment $[\![-]\!]^I : Fm \longrightarrow PF$ according to definition 3.5.

We proceed using a proof by induction on the complexity of $\mathcal{A}$ the claim that, *for every interpretation* I, $[\![\mathcal{A}]\!]_{f^I} = I(\mathcal{A})$. We show the cases for atomic formulas and universally quantified formulas, leaving the straightforward propositional connective cases to the reader.

Case $\mathcal{A} = P^n(x_{i_1}, \ldots, x_{i_n})$: Note the following:

$$\begin{aligned} I(P^n(x_{i_1}, \ldots, x_{i_n})) &= I(P^n)(I(x_{i_1}), \ldots, I(x_{i_n})) \\ &= M^I(P^n)(f^I(x_{i_1}), \ldots, f^I(x_{i_n})) \\ &= [\![P^n(x_{i_1}, \ldots, x_{i_n})]\!]_{f^I} \end{aligned}$$

This series of equations holds for each I, and therefore the base case is shown.

Case $\mathcal{A} = \forall x_k \mathcal{B}$:

$$\begin{aligned} I(\forall x_k \mathcal{B}) &= \textstyle\prod \{I^{b/x_n}(\mathcal{B}) : b \in D\} && \text{Definition} \\ &= \prod_{b \in D} I^{b/x_k}(\mathcal{B}) && \text{Notation} \\ &= \prod_{b \in D} [\![\mathcal{B}]\!]_{f^{I^{b/x_k}}} && \text{Induction Hypothesis} \\ &= \forall_k [\![\mathcal{B}]\!]_{f^I} = [\![\forall x_k \mathcal{B}]\!]_{f^I} \end{aligned}$$

The final step in this series of equalities needs some explanation. By the definition of $\forall_x$, if $\bigwedge_{f' \sim_n f} \varphi f'$ is in C, then $\forall_n \varphi f = \bigwedge_{f' \sim_n f} \varphi f'$. Brady has shown that the required generalized conjunction of instances in an element of C, and so the last equality above is justified, for the set of variable assignments in the second last line are just the set of x_k variants of f^I. □

Thus we end the proof of the main theorem of this section. We in addition obtain the following corollary.

Corollary 5.9. If $\mathcal{A}$ is not valid in the class of unreduced L-model structures, then $\mathcal{A}$ is not valid in the class of MG Matrices

6. From the Matrix, Content

Now for the other direction, we proceed from MG Matrices to unreduced L-model ("content") structures. This section will consist of a proof of the following:

Theorem 6.1. For any MG Model $\mathfrak{M}$ that is *rich enough* (defined below), there is an unreduced L-model structure $\mathfrak{S}^{\mathfrak{A}}$ such that, for every variable assignment f, there is an interpretation I_f on $\mathfrak{S}^{\mathfrak{A}}$ where $[\![\mathcal{A}]\!]_f \in \mathcal{D}$ iff $I_f(\mathcal{A}) \in T$.

In the previous section, much of the work was showing how to *simplify* content structures into the more spare machinery required by MG matrices. In this section we must do the opposite, namely, show that, if we start from MG matrices which are *rich enough*, then we can represent all the various moving parts of content structures. The task then is to, given a rich enough MG matrix $\mathfrak{A} = \langle A^{\mathfrak{N}}, A^{\mathfrak{C}}, D, PF, h\rangle$ define a corresponding content structure:

$$\mathfrak{S}^{\mathfrak{A}} = \left\langle C^{\mathfrak{A}}, T^{\mathfrak{A}}, D^{\mathfrak{A}}, \mathbb{F}^{\mathfrak{A}}, \leq^{\mathfrak{A}}, *^{\mathfrak{A}}, \sqcap^{\mathfrak{A}}, \sqcup^{\mathfrak{A}}, \Rightarrow^{\mathfrak{A}}, \textstyle\prod^{\mathfrak{A}}, \coprod^{\mathfrak{A}} \right\rangle$$

(going forward, we'll drop some of the superscripted $\mathfrak{A}$'s, where confusion due to ambiguities are unlikely to arise).

So we need to find, or build, appropriate objects for each of the content structure's elements. To start with, we can help ourselves simply to a set of contents $C^{\mathfrak{A}} = A^{\mathfrak{N}}$.[15] Similarly obvious is the choice $T^{\mathfrak{A}} = \mathcal{D}^{A^{\mathfrak{N}}}$. That is, we can just help ourselves to the contents, as well as the true contents, by taking the elements of our nugget, and the designated elements of the nugget, respectively. It is quite natural to just help ourselves to $D^{\mathfrak{A}} = D$. Finally, to finish the easy bits, we can just help ourselves to the content relations/operations by taking over the operations on $A^{\mathfrak{N}}$ in the obvious way (i.e., fixing $\Rightarrow^{\mathfrak{A}}$ to be $\rightarrow^{\mathfrak{N}}$). Taking these definitions over makes it fairly clear that the desired *propositional* structure is preserved, as the constraints on BB matrices can be seen to be equivalent to the constraints (p1)–(p7) on content structures. Similarly, in both cases the interpretations behave as homomorphisms from the language to the structure. So when considering just the propositional structure of formulas, it is clear that the interpretations will behave the same, and so it is easy to go from an MG matrix to a content structure.

As expected, the complexity comes in when we consider the behaviour of the quantifiers, and particularly the use Brady makes of the "content functions" $\mathbb{F}$, which are related to, but not quite the same as, the propositional functions PF of MG matrices. The general mismatch between the set $\mathbb{F}$ and the set of admissible propositions is what the represent. A propositional function represents formulas, while a function in $\mathbb{F}$ represents formulas without concrete occurrences of variables.

MG matrices are 'general' in the sense of having a distinguished set of *admissible* propositional functions PF, but these are functions which take simply a variable assignment to return a proposition. That is, for $\varphi \in PF$, the only thing we need to know is how to interpret the 'free variables' which 'occur' in φ in order to know what proposition it expresses. In Brady's structures, however, what $F \in \mathbb{F}$ takes as argument is an *ordered set* of *domain objects*, and there is a great deal of machinery introduced in order to ensure that we have enough such F's to represent every formula we may want to interpret.[16] In addition, we construct interpretations of complex formulas by a somewhat complicated procedure involving taking care of the elements of $\mathbb{F}$ interpreting the subformulas and the sets of domain objects *separately*. All this introduces a great deal of 'syntactic sugar' into Brady's modeling. In this sense, the 'general frame' flavour of Brady's structures is much stronger than is the case in MG matrices. As a result, if we want to use MG matrices to represent Brady structures, we need to consider a class of MG matrices with enough structure in order to be able to represent all of this syntax in Brady's semantics.

One interesting side effect of Brady's building in the syntax into the semantics in this way is that there can be curious mismatches between the richness of the language we want to model, and the richness of the structures we have to do the modeling. Note, e.g., that if our language has only a single unary predicate, Brady's structures would nonetheless have n-ary members of $\mathbb{F}$ for each n. For example, every finite conjunction of the function representing that unary predicate. MG matrices don't build in all this structure, and so in order to construct an element of $\mathbb{F}$ using only propositional functions, we must be able to

[15]In general, we can ignore $A^{\mathfrak{C}}$ and h, except insofar as they give $\forall_n$ and $\exists_n$ the desired properties, as given these assumptions we will see that $A^{\mathfrak{N}}$ is populated with *enough* meets and joins required for a content structure.

[16]For example, consider an atomic formula $P(x, y, z)$ with three free variables. In a sense, what represents this formula in the MG setting is a $\varphi^P \in PF$ which represents "$P(x, y, z)$", whereas what represents P in a Brady structure is a $F^P \in \mathbb{F}^3$, which needs to be embroiled in further machinery to represent the role of the variables x, y, z.

isolate n-ary (constructions of variable assignments and) propositional functions. Having enough propositional functions around to do this is precisely what we'll require of *rich enough* MG matrices, and models built thereon.[17]

To consider rich enough *models* let us fix a language to interpret. This gives us, in the MG Models, the function M which takes each n-ary predicate to an element of PF. (This will be the exact function type for elements of $\mathbb{F}$ in the corresponding Brady structure.) By using these functions, we obtain elements in $\mathbb{F}$ for each of the predicates. It is easy to see that we can obtain functions for complex formulas, if we can reduce them to predicates. We can do so under a fairly heavy assumption on our models/languages. We identity the following classes of MG Models.

Definition 6.2 (Rich Enough MG Models)**.** An MG model $\mathfrak{M}$ (with language $\mathfrak{L}$) is *rich enough* when for any formula $\mathcal{A}$ there exists a atomic formula $P^{\mathcal{A}}$ whose arity is the sum of constants and free variables in $\mathcal{A}$ such that $[\![\mathcal{A}]\!]^{\mathfrak{M}}_f = [\![P^{\mathcal{A}}(\vec{\tau})]\!]^{\mathfrak{M}}_f$, where $(\vec{\tau})$ is the constants and free variables that occur in $\mathcal{A}$ in the order they occur from left to right. We will use this notation for the corresponding predicate for a formula.

To take a concrete example, for the formula $\forall x P(x, y, z) \wedge R(c, x, z)$ would correspond to a predicate Q of arity 5 such that

$$[\![\forall x P(x, y, z) \wedge R(c, x, z)]\!]^{\mathfrak{M}}_f = M(Q)(f(y), f(z), f(c), f(x), f(z)) \ \ (= [\![Q(y, z, c, x, z)]\!]^{\mathfrak{M}}_f)$$

This correspondence enabled by the rich enough models will be key in constructing the context structures below. Rich enough models require, by analogy, a rich enough language.

Definition 6.3. For any rich enough MG Model $\mathfrak{M} = \langle \mathfrak{A}, M \rangle$ with $\mathfrak{A} = \langle A^{\mathfrak{N}}, A^{\mathfrak{C}}, D, PF, h \rangle$, define a set $\mathbb{F}(M)$ functions from D into $A^{\mathfrak{N}}$ (the carrier set of $A^{\mathfrak{N}}$) as M restricted to inputs of the form D^n.

It is clear, given the previous definition, that each element of $\mathbb{F}(M)$ is equivalent to the interpretation of a predicate letter $P^{\mathcal{A}}$ for some $\mathcal{A}$. That is, each element of $\mathbb{F}(M)$ is of the form $M(P^{\mathcal{A}})$.

Definition 6.4 (Corresponding BBQ m.s.)**.** Let $\mathfrak{M} = \langle \langle A^{\mathfrak{N}}, A^{\mathfrak{C}}, D, PF, h \rangle, M \rangle$ be a MG Matrix. The *corresponding Corresponding BBQ m.s.* $\mathfrak{S}^{\mathfrak{M}}$ is defined as follows:

(1) $C = A^{\mathfrak{N}}$.
(2) $T = \mathcal{D}$.
(3) $D = D$.
(4) $\mathbb{F} = \mathbb{F}(M)$.
(5) $\leq = \leq^{\mathfrak{N}}; * = \neg^{\mathfrak{N}}, \sqcap = \wedge^{\mathfrak{N}}; \sqcup = \vee^{\mathfrak{N}}; \Rightarrow = \rightarrow^{\mathfrak{N}}$.
(6) $\bigsqcap\{M(P^{\mathcal{A}})\ {}^{b/\langle \vec{j} \rangle_m}(\vec{a}_{n-m}) \mid b \in D\} = M(P^{\forall x_k \mathcal{A}[x_k/\langle j \rangle_m]})(\vec{a}_{n-m})$
(7) $\bigsqcup\{M(P^{\mathcal{A}})\ {}^{b/\langle \vec{j} \rangle_m}(\vec{a}_{n-m}) \mid b \in D\} = M(P^{\exists x_k \mathcal{A}[x_k/\langle j \rangle_m]})(\vec{a}_{n-m})$

The essential tension between Brady's modeling of formulas-sine-variable-occurrences and modeling formulas is highlighted in (6) and (7). In these, the occurrence of $\mathcal{A}$ on the left has particular terms in a particular order, which may clash with arbitrary n-length sequences of objects: e.g., take $\mathcal{A} = P(x, x)$ while the sequence given is (a, b). We thus ignore the actual

[17]Sufficiently rich MG matrices (minus models) ought to be capable of being transformed into Brady structures, but we choose to transform models because (1) any model on any MG structure can be transformed accordingly, and (2) MG models carry the syntactical information/coresopndence required to build a Brady Structure.

variables in $\mathcal{A}$ on the left. On the right, however, we must take the variable occurrences seriously. Thus, we write $\mathcal{A}[x_k/\langle j\rangle_m]$ as the formula $\mathcal{A}$ but with the terms occurring in the j-th places (left-to-right on free variables and constants) replaced with a variable k not occurring on $\mathcal{A}$ (to avoid further clashes).[18] Note also that in (6) and (7) we have defined the $\bigsqcap$ and $\bigsqcup$ as of type $V \longrightarrow C$, as required.

Note that the following two lemmas require us to already have correlated $\neg$ with $*$, $\sqcap$ with $\wedge$ and so forth.

Lemma 6.5 (Closure under $*, \sqcap, \sqcup, \Rightarrow$)**.** Let The set $\mathbb{F}(M)$ is closed under $*, \sqcap, \sqcup, \Rightarrow$: that is, closure conditions (c1) and (c2) are met by the defined set of content functions.

Proof. We show the case for $*$ – the other cases are similar. We assume that any variable assignment is surjective unless otherwise noted, and so we will not include the hat notation.

Case $*$: Let $M(P^{\mathcal{A}}) \in \mathbb{F}(M)$. Consider the function $M(P^{\neg\mathcal{A}})$ which is in $\mathbb{F}(M)$ by definition. We have:

$$\begin{aligned}
[M(P^{\mathcal{A}})(a_1, \dots a_n)]^* &= [[\![P^{\mathcal{A}}(\tau_{a_1}, \dots \tau_{a_n})]\!]_f]^* && \text{for some } f \in D^{\omega} \\
&= [[\![\mathcal{A}]\!]_f]^* && \text{by definition} \\
&= \neg[\![\mathcal{A}]\!]_f && \text{by } * = \neg^{\mathfrak{M}} \\
&= [\![\neg\mathcal{A}]\!]_f && \text{property of } \neg \\
&= [\![P^{\neg\mathcal{A}}(\tau_{a_1}, \dots \tau_{a_n})]\!]_f && \text{same } \vec{\tau} \\
&= M(P^{\neg\mathcal{A}})(a_1, \dots a_n) && \text{same } \vec{\tau} \text{ and } f
\end{aligned}$$

The step leading to the second line is justified by the defined correspondence between $\neg$ and $*$. In general, we have $M(P^{\mathcal{A}\otimes\mathcal{B}}) = M(P^{\mathcal{A}}) \otimes M(P^{\mathcal{B}})$. □

Lemma 6.6. The set $\mathbb{F}(M)$ satisfies condition (c3) in the definition of BBQ model structures, namely: if $M(P^{\mathcal{A}}) \in \mathbb{F}^n$ and $\langle\vec{j}\rangle_m \subseteq \{1, \dots, n\}$, then $\bigsqcap\{M(P^{\mathcal{A}})^{\,b/\langle\vec{j}\rangle_m} \mid b \in D\} \in \mathbb{F}^{n-m}$ and $\bigsqcup\{M(P^{\mathcal{A}})^{\,b/\langle\vec{j}\rangle_m} \mid b \in D\} \in \mathbb{F}^{n-m}$.

Proof. By definition, we remind the reader, we have that

$$\bigsqcap\{M(P^{\mathcal{A}})^{\,b/}\langle\vec{j}\rangle_m \mid b \in D\}(\vec{a}_{n-m}) =_{df} \bigsqcap\{M(P^{\mathcal{A}})^{\,b/}\langle\vec{j}\rangle_m(\vec{a}_{n-m}) \mid b \in D\}$$

for all $a_1, \dots, a_{n-m} \in D$. The right-hand-side of the identity is defined to be $M(P^{\mathcal{A}})(\vec{a}_{n-m})$, which is an element of $\mathbb{F}(M)$ by the assumption of the model being rich enough. Moreover, the function is of the correct arity: i.e., $n - m$. The case for $\bigsqcup$ is similar. □

Lemma 6.7. For any rich enough MG model $\mathfrak{M}$, the corresponding content structure $\mathfrak{S}^{\mathfrak{M}}$ is a BBQ structure.

Proof. It is clear from the definitions that $\mathfrak{S}^{\mathfrak{M}}$ is the right sort of thing to be a BBQ structure, in the sense that all the defined elements are of the correct type (note that the V stated in the typing of $\bigsqcap, \bigsqcup$ is implicit, but available, given that we have all n-ary functions in the rich enough model). So the remainder of the task concerns showing that the constraints on BBQ structures are satisfied.

[18] To make reading (6) and (7) easier, the reader is reminded that $F^{n\,b/\langle\vec{j}\rangle_m}(\vec{a}_{n-m}) = F^n[^{b/\langle\vec{j}\rangle_m}(\vec{a}_{n-m})]$.

Next, it is required to show that all of the conditions (p1)–(p11) are satisfied. As mentioned before, the cases (p1)–(p7) are straightforward, and left to the reader. Here we consider (p8), (p10), and (p11), as (p9) is very similar to (p8).

Towards (p8).(a), let us unpack the definition of $\bigsqcap$:

$$\begin{aligned}
&(1) & \bigsqcap\{M(P^{\mathcal{A}})f^{b/k}(\vec{i}_n) \mid b \in D\} &= \bigsqcap\{M(P^{\mathcal{A}})(f^{b/k}(\vec{i}_1), \ldots, f^{b/k}(\vec{i}_n)) \mid b \in D\} \\
&(2) & &= \bigsqcap\{M(P^{\mathcal{A}})^{b/}\langle\vec{j}\rangle_m(\vec{a}_{n-m}) \mid b \in D\} \\
&(3) & &= M(P^{\forall_k})(\vec{a}_{n-m}) \\
&(4) & &= [\![\forall_k \mathcal{A}]\!]^{\mathfrak{M}}_f \\
&(5) & &\leq [\![\mathcal{A}]\!]^{\mathfrak{M}}_f
\end{aligned}$$

where in line (2) and beyond, $\langle\vec{j}\rangle_m$ is the indexes of occurrences of k in $(\vec{i}_n)$ and the $(\vec{a}_{n-m})$ are the remaining $(f^{b/k}(i_1), \ldots, f^{b/k}(i_n))$. Equalities (1) and (2) are just changing the description of a single sequence of objects. (3) is the defining equality of the $\bigsqcap$. (4) follows from (3), and keeps the original f from above. (Although it could be changed for any f agreeing on all the terms in $\mathcal{A}$, the proof follows nicely using the same variable assignment.) (5) is a fact on MG models.

Two more equalities matter for this case:

$$\begin{aligned}
&(6) & [\![\mathcal{A}]\!]^{\mathfrak{M}}_f &= M(P^{\mathcal{A}})(f^{b/k}(\vec{i}_1), \ldots, f^{b/k}(\vec{i}_n)) \\
&(7) & &= M(P^{\mathcal{A}})\ f^{b/k}(\vec{i}_n)
\end{aligned}$$

(6) follows by the definition of *rich enough* models (with the terms being obtained from previous lines). (7) is a redescription of a sequence of objects.

For (p8).(b), suppose that $M(P^{\mathcal{A}})\ f(\vec{i}_n)$ is k-constant and that $M(P^{\mathcal{A}})\ f(\vec{i}_n) \leq M(P^{\mathcal{B}})\ f^{b/k}(\vec{j}_m)$ for each $b \in D$. Then:

$$\begin{aligned}
&(8) & M(P^{\mathcal{A}})\ f(\vec{i}_n) &\leq M(P^{\mathcal{B}})\ f^{b/k}(\vec{j}_m) \\
&(9) & M(P^{\mathcal{A}})\ f(\vec{i}_1), \ldots, f(\vec{i}_n)) &\leq M(P^{\mathcal{B}})\ f^{b/k}(\vec{j}_1), \ldots, f^{b/k}(\vec{j}_m)) \\
&(10) & [\![\mathcal{A}]\!]_f &\leq [\![\mathcal{B}]\!]_{f^{b/k}} \text{ each } b \in D \\
&(11) & [\![\mathcal{A}]\!]_f &\leq [\![\forall_k \mathcal{B}]\!]_f \\
&(12) & M(P^{\mathcal{A}})\ f(\vec{i}_n) &\leq \bigsqcap\{M(P^{\mathcal{B}})\ f^{b/k}(\vec{j}_m) \mid b \in D\}
\end{aligned}$$

The step from line (10) to (11) is justified by the fact that if $[\![\mathcal{A}]\!]_f \leq [\![\forall_k \mathcal{B}]\!]_{f^{b/k}}$ holds for each $b \in D$, then we have that $[\![\mathcal{A}]\!]_f \leq \bigwedge^{\mathfrak{C}}_{f' \sim_k f} [\![\mathcal{B}]\!]$, from which it follows by the definition of MG matrix that (4) holds.

The proofs for (a) and (b) of (p9) are similar, straightforward, and left to the reader.

As for (p10), assume that $M(P^{\mathcal{A}})$ is an n-ary function and $M(P^{\mathcal{B}})$ an m-ary function, and let $M(P^{\mathcal{A}})\ f^{b/k}(\vec{i}_n)$ be k-constant. By Lemma 6.5, we have that $M(P^{\mathcal{A}}) \sqcup M(P^{\mathcal{B}}) = M(P^{\mathcal{A} \vee \mathcal{B}})$, and so then

$$\begin{aligned}
\bigsqcap\{(M(P^{\mathcal{A}}) \sqcup (M(P^{\mathcal{B}}))f^{b/k}(\vec{i}_n, \vec{j}_m) \mid b \in D\} &= \bigsqcap\{M(P^{\mathcal{A} \vee \mathcal{B}})f^{b/k}(\vec{i}_n, \vec{j}_m) \mid b \in D\} \\
&= M(P^{\forall_k((\mathcal{A} \vee \mathcal{B})[x_k/\langle j\rangle_q])})(\vec{a}_{n+m-q})
\end{aligned}$$

Given the k-constancy assumption, k does not occur in $\vec{i}_n$, and so each replacement of x_k in $((\mathcal{A} \vee \mathcal{B})[x_k/\langle j\rangle_q])$ occurs in $\mathcal{B}$.

Now by the assumption the model is rich enough

$$M(P^{\forall_k((\mathcal{A}\vee\mathcal{B})[x_k/\langle j\rangle_q])})(\vec{a}_{n+m-q}) = [\![\forall_k((\mathcal{A}\vee\mathcal{B})[x_k/\langle j\rangle_q])]\!]_{f^{b/k}}$$
$$\leq [\![(\mathcal{A}\vee\forall_k\mathcal{B}[x_k/\langle j\rangle_q])]\!]_{f^{b/k}}$$

With this inequality in mind, we can obtain the desired inequality by unpacking the right-hand side further, as follows:

$$\begin{aligned}[\![(\mathcal{A}\vee\forall_k\mathcal{B}[x_k/\langle j\rangle_q])]\!]_{f^{b/k}} &= M(P^{(\mathcal{A}\vee\forall_k\mathcal{B})[x_k/\langle j\rangle_q])})(\vec{a}_{n+m-q})\\ &= (M(P^{\mathcal{A}})\sqcup M(P^{\forall_k\mathcal{B})[x_k/\langle j\rangle_q]})(\vec{a}_{n+m-q})\\ &= M(P^{\mathcal{A}})\ f^{b/k}(\vec{i}_n)\sqcup\bigsqcap\{M(P^{\mathcal{B}})f^{b/k}(\vec{j}_m)\mid b\in D\}\end{aligned}$$

The last equality packs in several steps, which we leave to the reader to confirm. The result is, as desired, that:

$$M(P^{\forall_k((\mathcal{A}\vee\mathcal{B})[x_k/\langle j\rangle_q])})(\vec{a}_{n+m-q}) \leq M(P^{\mathcal{A}})\ f^{b/k}(\vec{i}_n)\sqcup\bigsqcap\{M(P^{\mathcal{B}})f^{b/k}(\vec{j}_m)\mid b\in D\}$$

As for (p11), assume that $M(P^{\mathcal{A}})\ f^{b/k}(\vec{i}_n)\in T$ for all $b\in D$. Then $[\![\mathcal{A}]\!]_f^{b/k}\in\mathcal{D}$ for all $b\in D$. By condition (5).(d) of definition 3.4, it follows that $[\![\forall x_k\mathcal{A}]\!]_f\in\mathcal{D}$. But then, because h is a matrix homomorphism, translating back to the content structure we have that $\bigsqcap\{M(P^{\mathcal{A}})\ f^{b/k}(\vec{i}_n)\mid b\in D\}\in T$, as required. □

Lemma 6.8. For any rich enough MG model $\mathfrak{M}$ with interpretation $[\![\cdot]\!]$ (based on M) and variable assignment $f\in D^\omega$ on an MG matrix $\mathfrak{A}$, we can find an interpretation I^f on $\mathfrak{S}^{\mathfrak{A}}$ such that for any formula A:

$$[\![A]\!]_f = I^f(A)$$

Proof. Given the variable assignment f we begin the construction of the corresponding I^f as follows:

(1) $I^f(c) = M(c)$
(2) $I^f(x_n) = M_f(x_n)$
(3) $I^f(P^n) = M(P^n)$
(4) This is extended to formulas by definition 4.3.

Note that the corresponding interpretations are largely restricted by M and not the variable assignment. Using this interpretation we show the target by induction on the complexity of a formula.

If $\mathcal{A}$ is atomic, then $\mathcal{A} = P^n(\tau_1,\ldots\tau_n)$. Consider the following:

$$\begin{aligned}I^f(P^n(\tau_1,\ldots\tau_n)) &= I^f(P^n)(I^f(\tau_1),\ldots I^f(\tau_n))\\ &= M_f(P^n)(M_f(\tau_1),\ldots M_f(\tau_n))\\ &= [\![P^n(\tau_1,\ldots\tau_n)]\!]_f^m\end{aligned}$$

We leave the straightforward propositional connectives to the reader. The only case we show is $\mathcal{A} = \forall x_k\mathcal{B}$. We first note the equivalence between $I^{f^{b/k}}$ and $(I^f)^{b/k}$. We leave the proof of this equivalence to the reader, with the observation that the treatment of terms and predicate letters is the same. Using this equivalence obtain the first step in the following:

$$I^f(\forall x_k \mathcal{B}) = \bigsqcap\{I^{f^{b/k}}(\mathcal{B}) \mid b \in D\} \tag{13}$$

$$= \bigsqcap\{[\![\mathcal{B}]\!]_{f^{b/k}} \mid b \in D\} \tag{14}$$

$$= \bigsqcap\{M(P^{\mathcal{B}})\ f^{b/k}(\vec{\tau}_n) \mid b \in D\} \tag{15}$$

$$= M(P^{\forall x_k \mathcal{B}[x_k/\langle j\rangle_m]})(\vec{a}_{n-m}) \tag{16}$$

$$= [\![\forall x_k \mathcal{B}]\!]_f \tag{17}$$

The step (14) is justified by the induction hypothesis. Line (15) is justified by the MG model being rich enough, where the terms in $(\vec{\tau}_n)$ are the free variables and constants occurring in $\mathcal{B}$. In (16), the list $(\vec{a}_{n-m})$ is the objects $f^{b/k}(\vec{\tau}_n)$ minus the occurrences of x_k. The last step is again by the model being rich enough. □

We can combine these lemmas to obtain a proof of the following corollary, from which the corollary follows.

Corollary 6.9. If A is not valid in the class of MG matrices, then it is not valid in the class of BBQ structures.

Proof. This is a clear corollary of the previous lemma, provided it is shown that for every $\mathcal{A}$ not valid in the class of MG matrices, there is a *rich enough MG model* on which $\mathcal{A}$ is invalid.

Take a model $\mathfrak{M}_0$ in which a formula $\mathcal{A}$ is invalid. Take an enumeration of the formulas in the language $\mathfrak{L}_0$ of $\mathfrak{M}_0$. For each formula $\mathcal{B} \in \mathfrak{L}_0$, if there is a predicate equivalent, continue. If there is not, then add a new predicate $P^{\mathcal{B}}$ creating language $\mathfrak{L}_1 = \mathfrak{L}_0 \cup \{P^{\mathcal{B}}\}$. Moreover, set the new interpretation M_1 such that it remains as M_0 except that $[\![\mathcal{B}]\!]_f = M(P^{\mathcal{B}})\ (f(\tau_1), \ldots, f(\tau_n))$, where the τ are the (left-to-right) occurrences of free variables and constants in $\mathcal{B}$, as in the definition of rich enough models. This ensures that $[\![P^{\mathcal{B}}(\tau_1, \ldots \tau_n) \leftrightarrow \mathcal{B}]\!]_f$ is always designated.

It is more or less immediate that if $\mathcal{C}$ in $\mathfrak{L}_0$, it is designated in M_1 iff it is designated in M_0.

We repeat this step to obtain each $\mathfrak{M}_p$, in $\mathfrak{L}_p$ for each $p \in \omega$, and take the union of each language to be $\mathfrak{L}^0_\omega$. For each step, we obtain the new model that conservatively extends validity in $\mathfrak{L}_0$. We repeat this process ω-many times, increasing the superscript, and take the union of these languages as $\mathfrak{L}^\omega_\omega$. Similarly, we obtain $\mathfrak{M}^\omega_\omega$. This last model is rich enough, as can be shown by fairly routine arguments. □

7. Conclusion

In this paper, we've compared Brady's content structures (for BBQ) with MG matrices appropriate for BBQ. We've shown how to transform any of the former into one of the latter, and have pinned down a subset of the latter (the *rich enough* MG matrices) which can be transformed into the former. This serves to indicate the distinctive features of Brady's content semantics for quantified logics, by comparing them with the more recent, and in many ways simpler, semantics devised by Mares and Goldblatt (or a matricial variation thereon). Along the way, we've seen how much *syntactic* flavour Brady's content semantics has, and how this seems to necessitate starting from MG matrices which are rich enough to represent not only truth conditions for a language, but particular syntactic features of the language itself, in order to build content structures out of them. This highlights the way in which Brady's semantics works by straddling the syntax/semantics boundary.[19]

What we've shown falls short of a full *duality*, for which we'd need to show that composing these transformations defines the appropriate identities in the respective sets of structures. Furthermore, we've focused our attention just on BBQ, and have not addressed extensions/expansions of this logic beyond the usual connectives. We invite the interested reader to investigate these avenues, as well as potential connections to other semantics for quantified relevant logics. We hope that these initial results will be valuable for such avenues of investigation, but leave our work here for now.

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