CONTENT AND DEPTH REVISITED

SHAY ALLEN LOGAN

ABSTRACT. In this paper I compare depth relevance to what has come to be called *depth hyperformalism*. Ordinary formality, is typically taken to require closure under uniform substitutions. Depth hyperformality requires closure under depth substitutions—not-necessarily-uniform substitutions that are allowed to vary with depth. As it turns out, all depth hyperformal logics are depth relevant but not vice-versa.

So we're left to ask the following question: for the particular projects Ross Brady is engaged in, should it be depth relevance or depth hyperformalism that should set the pace? More to the point, I am interested in the following two claims:

- Logics of meaning containment are necessarily depth hyperformal.
- Logics of meaning containment are necessarily depth relevant.

The first entails the second. Brady seems to endorse the second. I argue in this paper that he should also endorse the first.

INTRODUCTION

Relevant logics, as shown already in 1959 by Belnap, have the variable-sharing feature: their provable conditionals always share an atom between antecedent and consequent. In 1984, Ross Brady proved that for weak-enough relevant logics, we can say something stronger: the atom being shared will occur at the same *depth*—which is to say, in the scope of the same number of conditionals.

In this paper I will compare depth relevance to what has come to be called depth *hyperformalism*. In contrast to ordinary formality, which is typically taken to require closure under *uniform* substitutions, depth hyperformality requires closure under *depth* substitutions—not-necessarily-uniform substitutions that are allowed to vary with depth. As it turns out, all depth hyperformal logics are depth relevant but not vice-versa.[1]

So we're left to ask the following question: for the particular projects *Ross Brady* is engaged in, should it be depth relevance or depth hyperformalism that should set the pace? More to the point, I'll be interested in the following two claims:

[1]In the most recent work coming out of this line of research, different flavors of hyperformalism have been distinguished, hence the appending of 'depth ' to 'hyperformalism' here.

- Logics of meaning containment are necessarily depth hyperformal.
- Logics of meaning containment are necessarily depth relevant.

The first entails the second. Brady seems to endorse the second. I'll argue in this paper that for the sorts of projects he's working on, he should also endorse the first.

The question is, in a perfectly innocuous sense, purely academic. After all, the logic (DJ) that Brady advocates for in his best-known work on meaning containment (see [2]) is in fact depth hyperformal, as we'll see below. In spite of this, the general point I'm making is worth bearing in mind. Folks engaged in the kinds of projects Brady is engaged in ought not take their having arrived at a depth relevant logic as evidence of having arrived at a good-enough logic. What's instead required is that they arrive at a depth hyperformal logic.

1. Setup

We define our language, $\mathcal{L}$, as follows:

Vocabulary:

- Atomic formulas: p_1, p_2, etc. We write At for the set of these.
- Connectives: $\neg$, $\wedge$, $\vee$, and $\rightarrow$

Grammar:

- Each atomic variable v is a(n atomic) formula.
- If A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, and $(A \rightarrow B)$.

The metavariable conventions implicitly established in the above definition will be used throughout the paper. Outermost parentheses are dropped as per the usual conventions. For $A \in \mathcal{L}$, we write $\mathsf{At}(A)$ for the set of atomic formulas that occur in A. Given a set $S \subseteq \mathsf{At}$, $\mathcal{L}(S) = \{A : \mathsf{At}(A) \subseteq S\}$.

Given a formula A and a specified occurrence of B as a subformula of A, we define the depth of that occurrence of B in A recursively as follows:

- The depth of the unique occurrence of A in A is 0.
- If the depth of a given occurrence of B in A is n, then the depth of the corresponding occurrence of B in $\neg A$, in $A \wedge A'$, in $A' \wedge A$, in $A \vee A'$, and in $A' \vee A$ is also n.
- If the depth of a given occurrence of B in A is n, then the depth of the corresponding occurrence of B in $A \rightarrow A'$ and in $A' \rightarrow A$ is $n + 1$.

A *uniform substitution* is a function $\sigma : \mathsf{At} \rightarrow \mathcal{L}$. Given such a function we extend it to a function $\sigma^+ : \mathcal{L} \rightarrow \mathcal{L}$ by the following clauses:

- $\sigma(\neg A) = \neg\sigma(A)$.
- $\sigma(A \wedge B) = \sigma(A) \wedge \sigma(B)$.
- $\sigma(A \vee B) = \sigma(A) \vee \sigma(B)$.

- $\sigma(A \to B) = \sigma(A) \to \sigma(B)$.

We will almost always conflate σ^+ and σ and call both of them 'σ', since not confusion will arise by doing so.

Given a set of sentences X, we say that X is formal when for all uniform substitutions σ, if $A \in X$, then $\sigma(A) \in X$ as well. Thus 'formalism' (as we will use the word here) simply amounts to closure under uniform substitution.

Write $\mathbb{N}$ for the set of natural numbers. A *depth substitution* is a function $\delta : (\mathsf{At} \times \mathbb{N}) \to \mathcal{L}$. Given such a function we extend it to a function—also called δ—$(\mathcal{L} \times \mathbb{N}) \to \mathcal{L}$ by the following clauses:

- $\delta(\neg A, n) = \neg\delta(A, n)$.
- $\delta(A \wedge B, n) = \delta(A, n) \wedge \delta(B, n)$.
- $\delta(A \vee B, n) = \delta(A, n) \vee \delta(B, n)$.
- $\delta(A \to B, n) = \delta(A, n + 1) \to \delta(B, n + 1)$.

Given a set of sentences X, we say that X is depth hyperformal when for all depth substitutions δ and numbers n, if $A \in X$, then $\delta(A, n) \in X$ as well. Thus just as formalism is closure under uniform substitution, depth hyperformalism is closure under depth substitutions.

Ross Brady pioneered the study of depth in relevance logics in [3]. The idea to use nonuniform (and in particular depth-varying) substitutions to study depth relevance was first raised in [6]. What I think is now becoming clear, and what I will in part be arguing for below, is that depth relevance is, to some extent, a sideshow. What's driving all the action in this area is really the closure under depth substitutions bit.

1.1. **Some Examples.** An extremely useful example of a depth substitution is the function g defined by $(p_i, n) \mapsto p_{2^i 3^n}$. To get a feel for depth substitutions, think about how g acts on a few well-known theorems of classical logic:

Example 1:

$$\begin{aligned} g(p_1 \to (p_2 \to p_1), 0) &= g(p_1, 1) \to g(p_2 \to p_1, 1) \\ &= p_6 \to (g(p_2, 2) \to g(p_1, 2)) \\ &= p_6 \to (p_{36} \to p_{18}) \end{aligned}$$

Example 2:

$$\begin{aligned} g(((p_1 \wedge (p_1 \to p_2)) \to p_2, 0)) &= g(p_1 \wedge (p_1 \to p_2), 1) \to g(p_2, 1) \\ &= (g(p_1, 1) \wedge g(p_1 \to p_2, 1)) \to p_{12} \\ &= (p_6 \wedge (g(p_1, 2) \to g(p_2, 2))) \to p_{12} \\ &= (p_6 \wedge (p_{18} \to p_{36})) \to p_{12} \end{aligned}$$

Example 3:

$$\begin{aligned} g(p_1 \to (p_1 \vee p_2), 0) &= g(p_1, 1) \to g(p_1 \vee p_2, 1) \\ &= p_6 \to (g(p_1, 1) \vee g(p_2, 1)) \\ &= p_6 \to (p_6 \vee p_{12}) \end{aligned}$$

Example 4:

$$\begin{aligned} &g(((p_1 \to p_2) \wedge (p_2 \to p_3)) \to (p_1 \to p_3), 0) = \\ &\quad g(((p_1 \to p_2) \wedge (p_2 \to p_3)), 1) \to g(p_1 \to p_3, 1) = \\ &\quad (g(p_1 \to p_2, 1) \wedge g(p_2 \to p_3, 1)) \to (g(p_1, 2) \to g(p_3, 2)) = \\ &\quad ((g(p_1, 2) \to g(p_2, 2)) \wedge (g(p_2, 2) \to g(p_3, 2))) \to (p_{18} \to p_{72}) = \\ &\quad ((p_{18} \to p_{36}) \wedge (p_{36} \to p_{72})) \to (p_{18} \to p_{72}) \end{aligned}$$

Note that in Example 1 and Example 2, g transforms the given theorem of classical logic into an obvious non-theorem. On the other in Example 3 and Example 4 g instead transforms it into a theorem with the same logical form as the theorem we started with. Note also that this tracks the containment relation between DJ and classical logic: the formulas in Example 1 and Example 2 are not theorems of DJ; the formulas in Example 3 and Example 4 are. It follows from our discussion that, while the set of theorems of classical logic is, of course, it is not depth hyperformal. We will see below that DJ, on the other hand, *is* depth hyperformal.

The function g has a natural 'inverse', $\hat{g}$ defined by $\hat{g}(p_{2^i3^n}) = p^i$. Note that this is a *uniform* substitution. The sense in which $\hat{g}$ is an inverse to g is given by the following lemma:

Lemma 1. *For all n, $\hat{g}(g(A, n)) = A$*

Proof. By induction on A. □

The following lemma will also improve the readability of a few of the results below:

Lemma 2. *Given an atomic formula p, a number n, and a formula A, all occurrences of p in $g(A, n)$ are at the same depth.*

Proof. Suppose p occurs in $g(A, n)$. Then p is in the range of $g(-, n)$ so has the form $p_{2^i3^{j+n}}$. But this is possible only if it occurred at depth j in A. □

Our final observation is that formulas (like the ones examined in Example 1 and Example 2) that behave badly around depth substitutions tend to in fact behave so badly that they can't be included in anything even remotely like a logic. To see this in the case of Example 1, choose an arbitrary formula B, set of formulas X, and $A \in X$. Let d_B be a depth substitution for which the following hold:

$$d(p_1, 1) = A$$
$$d(p_2, 2) = A$$
$$d(p_1, 2) = B$$

Applying any such d_A to $p_1 \to (p_2 \to p_1)$, the result is $A \to (A \to B)$. Thus, if X is depth hyperformal, closed under modus ponens, and contains $p_1 \to (p_2 \to p_1)$ then X must also contain B. But B was chosen arbitrarily, so it in fact follows that every nonempty depth hyperformal set closed under modus ponens is either trivial or fails to contain $p_1 \to (p_2 \to p_1)$.

A similar trick works in the case of Example 2, though here we need to also assume the logic is closed under adjunction. In both cases, the moral is the same: no set of formulas can contain formulas like the ones in Example 1 and Example 2 while being both depth hyperformal and even remotely logic-like.[2]

2. Depth Friendliness

In this section we will describe a general procedure for showing that a given logic is depth hyperformal. The procedure is strictly more general than what is needed for the task at hand. All we really need to see is that DJ is depth hyperformal. Nonetheless, I think that since the effort it takes to gain the additional generality is quite minimal, and since the additional generality de-mystifies some parts of the proof that DJ is depth hyperformal, it's worthwhile to see the more general result.

We begin with a few definitions. An axiom scheme will be said to be depth friendly when the set of its instances is depth hyperformal. That is, an axiom scheme S is depth friendly when, given any instance A of S, any depth substitution d and any number n, $d(A, n)$ is again an instance of S. In a similar way, we say that the rule R is depth friendly when, given any instance $A_1, \ldots, A_n \Rightarrow B$ of R, any depth substitution d, and any number k, there are depth substitutions $d_1, \ldots, d_n$ and numbers $k_1, \ldots k_n$ so that $d_1(A_1, k_1), \ldots, d_n(A_n, k_n) \Rightarrow d(B, k)$ is again an instance of R. We say that a set of sentences X is depth friendly when X can be axiomatized by a set of depth friendly axiom schemes and a set of depth friendly rules.

Theorem 3. *If X is depth friendly, then X is depth hyperformal.*

[2]As is always the case with claims of this strength, there really ought to be caveats. One interesting class of caveats concerns empty logics of the sort examined in e.g. [5] or [9]. In [redacted] it is shown that depth substitutions in fact act on proofs. It thus seems likely that considerations like those above extended even to theoremless (or otherwise 'empty') logics, but I haven't taken the time to see whether this is correct.

Proof. By induction on derivations in X's depth friendly axiomatization.[3] To begin, choose a depth substitution d and a number n. A derivation of length one is just an instance, A of one of our axiom schemes S. By assumption, the axiom schemes are depth friendly. So $d(A, n)$ is again an instance of S. So it is also a derivation of itself and thus in X.

Now suppose we've ended our derivation by applying the following instance of the rule R:

$$A_1, \ldots, A_n \Rightarrow B.$$

We want to show that $d(B, n)$ is in X. By assumption, corresponding to d and n are depth substitutions $d_1, \ldots, d_n$ and numbers $k_1, \ldots k_n$ so that the following is again an instance of R:

$$d_1(A_1, k_1), \ldots, d_n(A_n, k_n) \Rightarrow d(B, n).$$

By the inductive hypothesis $d_1(A_1, k_1)$, …, and $d_n(A_n, k_n)$ are in X. Thus $d(B, n)$ is in X too. □

Fitting your definitions to your theorem is a practice which, when done in moderation, can be tolerable. What we have here is something slightly worse: the definitions have been fit not only to the theorem, but to its proof.

Such behavior is usually unforgivable. In this case, surprisingly, it bears fruit.[4] To see this, we first need to see that depth substitutions can be 'shifted' in ways that will turn out to be helpful. The following lemma explains what this means:

Lemma 4. *Let δ be a depth substitution and define δ' as follows:*

$$\delta'(p, n) = \begin{cases} \delta(p, n-1) & \text{if } n > 0 \\ \delta(p, 0) & \text{otherwise} \end{cases}$$

Then for all A and all $n > 0$, $\delta'(A, n) = \delta(A, n-1)$.

Proof. By induction on A. The definition gives us what we need for atoms and the inductive hypothesis immediately finishes the job for negations, conjunctions, and disjunctions. For conditionals, let $n > 0$ and note that $\delta'(A \to B, n) = \delta'(A, n+1) \to \delta'(B, n+1)$. By the inductive hypothesis, this is $\delta(A, n) \to \delta(B, n) = \delta(A \to B, n-1)$. □

In [2] Brady argues that (at the propositional level) the logic of meaning containment is the logic DJ.[5] DJ, in turn, is defined to be the smallest set

[3]The concept of depth friendliness was essentially cooked up to make this proof work, so it's no surprise that it does. Nonetheless, it's worth seeing it in action.

[4]Worth remark: one lesson of the parable of the garden is that not all things that bear fruit are forgivable.

[5]He actually argues that it's DJ^d, but then shows, in Corollary 2d to Theorem 4.13 that DJ and DJ^d are, qua sets of sentences, identical. They may (I'm genuinely uncertain) come

that (a) contains all uniform substitutions of the following formulas which is also (b) closed under all uniform substitutions of the following rules:

Axioms:

A1. $p_1 \to p_1$
A2. $(p_1 \wedge p_2) \to p_1$
A3. $(p_1 \wedge p_2) \to p_2$
A4. $((p_1 \to p_2) \wedge (p_1 \to p_3)) \to (p_1 \to (p_2 \wedge p_3))$
A5. $p_1 \to (p_1 \vee p_2)$
A6. $p_2 \to (p_1 \vee p_2)$
A7. $((p_1 \to p_3) \wedge (p_2 \to p_3)) \to ((p_1 \vee p_2) \to p_3)$
A8. $(p_1 \wedge (p_2 \vee p_3)) \to ((p_1 \wedge p_2) \vee (p_1 \wedge p_3))$
A9. $\neg\neg p_1 \to p_1$
A10. $(p_1 \to \neg p_2) \to (p_2 \to \neg p_1)$
A11. $((p_1 \to p_2) \wedge (p_2 \to p_3)) \to (p_1 \to p_3)$

Rules:

R1. $\dfrac{p_1 \qquad p_1 \to p_2}{p_2}$

R2. $\dfrac{p_1 \qquad p_2}{p_1 \wedge p_2}$

R3. $\dfrac{p_1 \to p_2 \qquad p_3 \to p_4}{(p_2 \to p_3) \to (p_1 \to p_4)}$

Lemma 5. DJ *is depth friendly.*

Proof. Each of the axiom schemes is clearly depth friendly. R2 is clearly depth friendly as well. So it comes down to R1 and R3.

R3 is easier: choose d and n and note that the following is an instance of R3:

$$\frac{d(p_1 \to p_2, n+1) \qquad d(p_3 \to p_4, n+1)}{d((p_2 \to p_3) \to (p_1 \to p_4), n)}$$

For R1, the only source of difficulty is that we need to use the depth substitution d' defined in Lemma 4. Using this, we see that the following is an instance of R1:

$$\frac{d(p_1, n) \qquad d'(p_1 \to p_2, n)}{d(p_2, n)}$$

□

apart if considered in other ways (e.g. if one were to somehow present them via something like the bunched proof systems given in [10]), but that's neither here nor there for the purposes of the present paper. Restricting our attention to DJ rather than DJ^d will let us ignore the somewhat fussy matter of disjunctive metarules. A referee has helpfully pointed out that we should be clear here: I am restricting my attention to logics-qua-set-of-theorems rather than logics-thought-of-some-other-way largely because it's what Brady does, and not because of anything particularly deep.

Corollary 6. DJ *is depth hyperformal.*

Proof. Immediate from the previous lemma and Lemma 3. □

Corollary 7. $A \in \mathsf{DJ}$ *iff* $g(A,0) \in \mathsf{DJ}$.

Proof. The 'only if' direction is immediate from the previous corollary. For the 'if' direction, recall from Lemma 1 that $A = \hat{g}(g(A,0))$. Thus, since $\hat{g}$ is a *uniform* substitution, it follows even without the previous corollary that if $g(A,0) \in \mathsf{DJ}$, then $A \in \mathsf{DJ}$. □

We also have the following as a corollary of the well-known main result in [1]:

Theorem 8. *If* $A \to B$ *is in* DJ *then there is an atom* p_i *so that* p_i *occurs in* A *and* p_i *occurs in* B.

We can then prove depth relevance for DJ by a surprisingly quick argument:

Theorem 9. *If* $A \to B$ *is in* DJ*, then there is an atom* p_i *and an* $n \in \mathbb{N}$ *so that* p_i *occurs at depth* n *in* A *and* p_i *occurs at depth* n *in* B.

Proof. By Theorem 6, $g(A \to B, 0) = g(A,1) \to g(B,1) \in \mathsf{DJ}$. Thus by Theorem 8, $g(A,1)$ and $g(B,1)$ share an atom. Since it occurs in the range of g there are $i \geq 1$ and n so that the shared atom is $p_{2^i 3^n}$. But also by construction, since $p_{2^i 3^n}$ occurs in $g(A,1)$, p_i must occur at depth $n-1$ in A. Similarly since $p_{2^i 3^n}$ occurs in $g(B,1)$, p_i must occur at depth $n-1$ in B. So p_i occurs at the same depth in both A and B as required. □

The chain of inferences we have, then, is something like this:

depth friendly ⇒ depth hyperformal ⇒ depth relevant (if relevant)

The converse of the second of these inferences is false; as mentioned in [7], this was noticed by Tore Fjeltland Øgaard. The converse to the first is, of course, true: one can simply take the class of all formulas as one's axioms.

3. Meaning Containment and Its Logic

It follows from Theorems 6 and 9 that the logic that is (at least one time-slice of-) Brady's preferred logic for a theory of meaning containment is in fact both depth hyperformal and depth relevant. But as pointed out above, knowing which of these two properties is the proper target of his inquiry is nonetheless important. After all, it is not as if Brady has finished plumbing the depths of his topic; further investigation of what meaning containment demands of us may well be afoot even now. Such investigations are served by having well-enunciated necessary syntactic criteria in play to (partially) judge their success by. The claim I'm making (and, in this section, defending) is that depth hyperformalism is a better syntactic criterion to use to this end than depth relevance is.

The argument I will present below can be summarized as follows:

- Nothing in Brady's meaning-containment semantics hinges on our assigning the same content to instances of a formula that occur at different depths.
- A depth varying version of Brady's content semantics, while formally more awkward, can help us make better sense of what Brady has to say about why his logic should be expected to be depth relevant.
- But it is depth hyperformalism and not depth relevance that is the obvious syntactic analogue to the depth varying assignments at the heart of the depth varying version of content semantics.

We begin with a brief overview of content semantics is it appears in [2]. But, in hopes of preventing typographical errors, we will dispense with one of Brady's conventions: rather than naming everything in sight with some variant or other of the letter 'c', we'll avail ourselves of a bit more of the alphabet. The result (while not different in any non-typographical way from what Brady does) is, I think, much easier on the eyes.

With that said, here are the details:

A content model structure is a 5-tuple $\langle K, T, *, \sqcup, c\rangle$ where K is a set of sets closed under intersection (with each member of K being a 'kontent'), $T \subseteq K$ is the set of true kontents, $\star$ is a unary function on kontents, and both $\sqcup$ and c are functions $K \times K \to K$.[6]

The intuition to have is that each 'kontent' is a possible content some sentence might have; T picks out those kontents that are the kontents of true sentences; and $\sqcup$ is the closed union of kontents. The so-called Routley star '$*$' is a tool used for dealing with negation. It's gone in for a lot of discussion over the years. We won't bother to say another word about it, though, because it simply plays no role at all in the arguments below. Finally, the binary function c is the function that assigns to a pair of kontents k_1 and k_2 the kontent that represents the content of the claim 'k_1 is contained in k_2'.

We require that these satisfy the following postulates:

p1. $k_1 \sqcup k_2 \supseteq k_1$; $k_1 \sqcup k_2 \supseteq k_2$.
p2. If $k_1 \supseteq k_2$ and $k_1 \supseteq k_3$, then $k_1 \supseteq k_2 \sqcup k_3$.

[6]I haven't been able to locate anywhere in Brady's corpus where the restriction to sets of kontents that are closed under intersection is made explicit. But it's clearly necessary else the clause for disjunctions in the semantics doesn't make sense. Brady also says that the function I'm here calling c is defined 'on containment sentences' or sometimes 'on containments'. Both of these strike me as odd, for a variety of reasons, but the main one is that neither way of understanding what c is would leave it an actual, genuine binary function from contents to contents, which is what its role in the semantic theory requires it be.

p3. $k^{**} = k$.
p4. If $k_1 \supseteq k_2$, then $k_2^* \supseteq k_1^*$.
p5. If $k_1 \supseteq k_2$ and $k_1 \in T$, then $k_2 \in T$.
p6. If $k_1 \in T$ and $k_2 \in T$, then $k_1 \sqcup k_2 \in T$.
p7. If $k_1 \sqcap k_2 \in T$, then $k_1 \in T$ or $k_2 \in T$.
p8. $c(k_1, k_2) \sqcup c(k_2, k_3) \subseteq c(k_1, k_3)$.
p9. $c(k_1, k_2) \sqcup c(k_1, k_3) \subseteq c(k_1, k_2 \sqcup k_3)$.
p10. $c(k_1, k_3) \sqcup c(k_2, k_3) \subseteq c(k_1 \sqcap k_2, k_3)$.
p11. $c(k_1, k_2) \supseteq c(k_2^*, k_1^*)$.
p12. $c(k_1, k_2) \in T$ iff $k_1 \supseteq k_2$
p13. If $k_1 \supseteq k_2$, then for all $k_3 \in K$, $c(k_3, k_1) \supseteq c(k_3, k_2)$ and $c(k_2, k_3) \supseteq c(k_1, k_3)$.

An interpretation then assigns a member $I(p)$ of K to each atom p and is extended to arbitrary sentences via the following clauses:

- $I(\neg A) = I(A)^*$.
- $I(A \wedge B) = I(A) \sqcup I(B)$.
- $I(A \vee B) = I(A) \sqcap I(B)$.
- $I(A \rightarrow B) = c(I(A), I(B))$.

To round things out, we say that a formula A is valid—and we write $\vDash A$—just if $I(A) \in T_M$ in all content model structures M and for all interpretations I.

Theorem 10. *The set of valid formulas is exactly the set of sentences in* DJ.

Proof. See [2]. □

As defined here, each atomic formula is assigned a kontent, which it retains throughout the evaluation of a formula. But I said above that nothing in the semantics depended on this. To see this, we present an alternative version of Brady's semantics that differs in only three ways:

- First, rather than using a single interpretation I we will instead rely on what I call *varying* interpretations, which are infinite families of interpretations $\{I_j\}_{j=0}^{\infty}$.
- Second, the semantic clause for the conditional is changed slightly:
$$I_j(A \rightarrow B) = c(I_{j+1}(A), I_{j+1}(B))$$
- Finally, we say that a formula is valid on varying assingments—and we write $\vDash_v A$—just if $I_0(A) \in T_M$ for all content model structures M and for all varying interpretations $\{I_j\}_{j=0}^{\infty}$.

Theorem 11. $\vDash A$ *iff* $\vDash_v g(A, 0)$

Proof. Note that by Corollary 7 and Theorem 10, $\vDash A$ iff $\vDash g(A, 0)$. So it suffices to see that $\vDash_v g(A, 0)$ iff $\vDash g(A, 0)$. To that end, for each $p \in$

$\mathsf{At}(g(A,0))$, let d_p be the (unique!) depth at which p occurs in $g(A,0)$. Given a content model structure M and a varying assignment $\{I_j\}_{j=0}^{\infty}$, let I be the assignment defined by $I(p) = I_{d_p}(p)$. I claim that if B occurs at depth n in $g(A,0)$, then $I(B) = I_n(B)$. To see that this will suffice, note that since $g(A,0)$ occurs at depth 0 in $g(A,0)$, $I(A) = I_0(g(A,0))$, and thus A is valid on constant assignments iff $g(A,0)$ is valid on varying assignments.

We prove the claim by induction on the complexity of B. If $B = p$ is an atom, then it occurs in $g(A,0)$ at depth d_p, and by definition $I(p) = I_{d_p}(p)$.

Suppose $B = B_1 \wedge B_2$ occurs at depth n in $g(A,0)$. By the inductive hypothesis, $I(B_1) = I_n(B_1)$ and $I(B_2) = I_n(B_2)$. So $I(B) = I(B_1) \sqcup I(B_2) = I_n(B_1) \sqcup I_n(B_2) = I_n(B)$. Similar arguments suffices for the negation and disjunction cases.

Suppose $B = B_1 \to B_2$ occurs at depth n in $g(A,0)$. By the inductive hypothesis, $I(B_1) = I_{n+1}(B_1)$ and $I(B_2) = I_{n+1}(B_2)$. So $I(B) = c(I(B_1), I(B_2)) = c(I_{n+1}(B_1), I_{n+1}(B_2)) = I_n(B)$. □

Theorem 12. *Let M be a content model structure, $\{I_j\}_{j=0}^{\infty}$ be a varying interpretation, and d be a depth substitution. Define the varying interpretation $\{I'_j\}_{j=0}^{\infty}$ by $I'_j(p) = I_j(d(p,j))$. Then in fact $I'_j(A) = I_j(d(A,j))$ for all formulas A.*

Proof. By induction on A. The only interesting case is the conditional case, for which we reason as follows:

$$\begin{aligned} I'_j(A \to B) &= c(I'_{j+1}(A), I'_{j+1}(B)) \\ &= c(I_{j+1}(d(A, j+1)), I_{j+1}(d(B, j+1))) \\ &= I_j(d(A, j+1) \to d(B, j+1)) \\ &= I_j(d(A \to B, j)) \end{aligned}$$

□

Corollary 13. *If $\vDash_v A$, then for all depth substitutions d, $\vDash_v d(A,0)$.*

Proof. Let $\vDash_v A$. Choose content model structure M and varying interpretation $\{I_j\}_{j=0}^{\infty}$. Let I' be as in the previous theorem. Then $I_0(d(A,0)) = I(A,0) \in T_M$ because $\vDash_v A$. So $\vDash_v d(A,0)$. □

Corollary 14. $\vDash_v g(A,0)$ *iff* $\vDash_v A$.

Proof. The 'if' direction is immediate from the preceding corollary. For the 'only if' direction, let $d(p,n) = \hat{g}(p)$ for all n and use the previous corollary again. □

Corollary 15. $\vDash A$ *iff* $\vDash_v A$.

Proof. Immediate from Theorem 11 and the preceding corollary. □

4. Depth Hyperformalism vs. Depth Relevance

Let's summarize what we've seen so far. In Corollary 6, we saw that DJ was depth hyperformal. Corollary 15 gave a sort of semantic analogue of this: content model structures can assign contents piecemeal, depth by-depth, without in any way changing the set of valid formulas. I *think* that Brady had an inkling of something at least quite like this result in mind. I think this for two reasons: first, something very like it is afoot in his [3], so it would be quite surprising if nothing in this neighborhood was on his mind at all by the time he got around to writing [2]. Second, something quite like the piecemeal approach seems to be at least in the air in the following passages:

> Depth relevance is an appropriate condition for a logic to satisfy since its relevance would then be ensured through a sentential variable that occurs at the same level on both sides of the main entailment '$\to$' thus enabling these sentential variables to interact with one another.[2, p. 164]
>
> a necessary condition for a subformula B to interact with another subformula B is for them to be on the same level, where level is counted down by '$\to$'s from the whole formula. For otherwise, the class of contents or containment statements involving the two B's will not match up. Such a concept of level can be elucidated as that of the *depth of a subformula occurrence of B in a formula A*[.][2, p. 162]

The idea seems to be this: when considering two occurrences of the same formula as a subformula of a given formula, if they occur at different depths, then they will be involved in different numbers of content containment statements. And this, in turn, means they can't 'interact'.

This is helpful, if a bit loose. But I think we can tie it to the above observations in two steps. First, recall that the binary function 'c' in a content model structure is meant to map each pair of kontents k_1 and k_2 to the kontent corresponding to *the claim that* k_1 is contained in k_2. The point of the italics here is to emphasize that we're bumping up against a fairly subtle bit of use-mention business.

Think for the moment about the simplest case and compare the formulas p and $p \to q$. Given a content model structure and an interpretation, the former is directly interpreted as one of the kontents. Suppose, for concreteness, that we have $I(p) = \{1, 2\}$ and $I(q) = \{2, 3\}$. Unlike p, $p \to q$ isn't directly interpreted as a kontent. Instead, it is interpreted as the kontent corresponding to the claim that $\{1, 2\} \subseteq \{2, 3\}$.

I want to be clear that I have no intuition at all about what kontent should correspond to such a claim. Importantly, however, this means that I have no particular reason to believe that it ought to be interpreted as something that in any way contains the kontent corresponding to p—that is, there's no reason to think that that kontent corresponding to the claim that $\{1, 2\} \subseteq \{2, 3\}$ must contain $\{1, 2\}$.

There is, then, a 'degree of freedom' introduced by the use of the 'kontent of the containment claim' function c. This freedom, in turn, is what keeps variables at different levels from, as Brady puts it, 'interacting'. This makes sense: since variables at different levels have *some* freedom from each other, they can differ. Thus, when we are doing logic, we must assume they *do* differ and see how much we can still get.

The problem, of course, is that while it seems clear from what's been said that the 'c' function introduces some amount of freedom, it's not totally clear how to precisely describe the amount of freedom it brings. After all, it's not as if this function is allowed to behave however it pleases—almost half of the semantic postulates restrict its behavior in one way or another. The point of the varying semantics introduced above was to quantify exactly how much freedom it introduces. The answer, we saw, was this: it introduces sufficient freedom to allow interpreting formulas occurring at different levels as if they were completely different formulas. And this, it seems clear, is better captured by requiring depth hyperformalism—which is very nearly a homophonic translation of the degree-of-freedom bit as explicated by the varying semantics into purely syntactic terms—than it is by depth relevance.

Conclusion and Future Work

I've argued that it is depth-hyperformalism and not depth relevance that is the right condition to require of logics of meaning containment. The heart of the argument I gave was this: Brady's own arguments for why we should expect a logic of meaning containment to be depth relevant appeal to the idea that depth might change meaning. But this is best captured by a varying semantics. And the obvious syntactic analogue to depth-varying content semantics is depth-hyperformalism and not depth relevance.

Future Work. Future work on content semantics ought to be constrained not by requiring depth relevance but by requiring depth substitution invariance. Elsewhere, this has been called 'hyperformalism'. But as it becoming clear (see e.g. [4], [11], and [8]), hyperformalism comes in a variety of flavors. One wonders how far the above arguments generalize in Brady's case.

It seems clear the arguments just given do in fact go a bit further than described, though care needs to be taken. As an example: it's not just entailments that have their value computed by some non-lattice function. Negations do too. This would seem, a priori, to also introduce a degree of freedom. But as shown in [4], if we take follow this line to its natural conclusion, we end up at either B or BM, depending on the decisions made along the way. In either case, we lose axiomatic contraposition and thus end up at a logic strictly weaker than DJ.

So the degree of freedom afforded by the semantics Brady provides for negation is less than the degree of freedom afforded by his semantics for the conditional. Naïvely it's unclear why this would be the case. One is tempted to lay the blame on the Brady-an semantic postulates concerning the star (p3, p4, and p11). But this doesn't quite work—there are even more postulates concerning the function c, but the freedom it allows is unrestricted. Thus, some investigation of degrees of freedom allowed by the various functions seems in order.

In a slightly different vein, it's also quite natural to think that the contribution made by the antecedent of a conditional should differ—and, in Brady's semantics, intuitively does differ—from the contribution made by the consequent of a conditional. But without a great deal of finesse, recognizing those degrees of freedom is quite destructive. After all, we probably want to preserve $p \to p$ as an axiom.

This leads to my final conclusion: there's a good deal more to be learned from content semantics. Brady's had the first few words on it. They're unlikely to be the last.

References

[1] Nuel D. Belnap. Entailment and relevance. *Journal of Symbolic Logic*, 25(2):144–146, 1960.

[2] Ross Brady. *Universal Logic*. CSLI Publications, 2006.

[3] Ross T. Brady. Depth relevance of some paraconsistent logics. *Studia Logica*, 43(1-2):63–73, 1984.

[4] Thomas Macaulay Ferguson and Shay Allen Logan. Topic transparency and variable sharing in weak relevant logics. *Erkenntnis*, pages 1–28, 2023.

[5] Rohan French. Structural reflexivity and the paradoxes of self-reference. *Ergo: An Open Access Journal of Philosophy*, 3, 2016.

[6] Shay Allen Logan. Depth relevance and hyperformalism. *Journal of Philosophical Logic*, 51(4):721–737, 2022.

[7] Shay Allen Logan. Correction to: Depth relevance and hyperformalism. *Journal of Philosophical Logic*, 52(4):1235–1235, 2023.

[8] Shay Allen Logan and Blane Worley. Hyperformalism for bunched natural deduction systems. *arXiv preprint arXiv:2409.10418*, 2024.

[9] Federico Pailos. Empty logics. *Journal of Philosophical Logic*, 51(6):1387–1415, 2021.

[10] Stephen Read. *Relevant Logic: A Philosophical Examination of Inference*. Blackwell, Oxford, 1988.

[11] Shawn Standefer, Shay Allen Logan, and Thomas Macaulay Ferguson. Topics, non-uniform substitutions, and variable sharing. *arXiv preprint arXiv:2409.08942*, 2024.