On the irrationality of the square root of 2*

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Abstract

This paper presents a proof of the irrationality of $\sqrt{2}$ which not only avoids use of paradoxes of implication but also eschews the principle of contraction. The actual theorem proved, in the relevant arithmetic $\mathbf{B}^\sharp$, is

$$\sim\exists x\exists y(x'\cdot x' = 2\cdot y\cdot y)$$

which is the theorem of natural number theory standardly expressing the irrationality of $\sqrt{2}$. The key move in the argument is to use provable cases of the law of the excluded middle

$$P \vee \sim P$$

or its close relative

$$P \vee (P \to (0 > 0))$$

to mimic the effect of contraction.

1 Introduction

It is the quixotic purpose of this paper to prove that there is no rational number whose square is 2. The non-quixotic part of the enterprise is to examine in detail the argumentative moves required for the result. While the irrationality of $\sqrt{2}$ is by now losing its once formidable power to shock and confuse mathematicians, the proof of that irrationality borders on interestingly uncharted regions in which lurk shocks a-plenty; and anyone not confused yet has only to read on.

Consider first a fairly tidy version of the ancient proof. The first move is to transform '$\sqrt{2}$ is irrational' into a statement of *natural* number theory:

> $\sqrt{2}$ is rational iff there are positive integers a and b such that
>
> $$\frac{a}{b} \times \frac{a}{b} = 2$$
>
> that is iff
>
> $$\frac{a^2}{b^2} = 2$$

*This paper is compiled from a manuscript '*Irrationals without Irrationality*' and a typescript '*The Square Root of 2 is Irrational (and no Funny Business)*' both dating from 1982 or early 1983. The manuscript is annotated "first draft rough" and the typescript is only a fragment, so the material has been considerably edited for present purposes. Given the progress in the field of substructural logic over the intervening 40 years, if I were to write the paper now, it would be differently expressed and presented, but I have chosen to retain the original as far as possible.

which is to say

$$a^2 = 2b^2.$$

Now we show by induction on a that there are no such numbers.

Base case: $a = 1$. For positive b, $2b^2 > 1$, so $a^2 \neq 2b^2$.
Hypothesis: For every $x < a$ and for every y, $x^2 \neq 2y^2$.
Induction case: suppose $a^2 = 2b^2$.

Then a^2 is even; so a is even
(since if a were odd, a^2 would be odd).
That is, for some x, $a = 2x$.
Therefore, for the same x, $a^2 = 4x^2$.
Applying this to the supposition, $4x^2 = 2b^2$.
Therefore $b^2 = 2x^2$;
But clearly $b < a$, contradicting the hypothesis.

This is, in fact, shorter and more elegant than the usual presentations which for some reason generally proceed by the lemma that any fraction may be expressed in its lowest terms (so a and b are assumed relatively prime) and goes on to show that a and b are both divisible by 2, contradicting the more complicated hypothesis.

There is no need at this point to spell things out in more formal detail, and for present purposes we gloss over the issues surrounding the reduction from rational to natural arithmetic. The goal of this paper is to reconstruct the proof that $a^2 = 2b^2$ has no solutions in the positive integers, using only the meagre logical resources provided by the basic relevant logic **B**. This exercise raises many difficulties: for instance, the inference from $a = 2x$ to $a^2 = 4x^2$ in invalid in weak relevant arithmetics, and that from $4x^2 = 2b^2$ to $b^2 = 2x^2$ is invalid even in the much stronger arithmetic $\mathbf{R}^\sharp$ [8]. Almost every line of the proof, in fact, requires non-trivial reconstruction. The key point to note, however, is that the crucial move is from

$$a^2 = 4x^2$$

to

$$2b^2 = 4x^2$$

in which the assumption

$$a^2 = 2b^2,$$

which has already been used to obtain the premise, is used again. Thus the proof reduces to absurdity that supposition *taken twice*, while the theorem is that the supposition is absurd on its own (taken once). The key logical shuffle, then, is that which takes us from

$$A, A \vdash B$$

to

$$A \vdash B.$$

This is the mysterious principle of *contraction* or *absorption* which is at the heart of almost everything deep and devious in logic. The aim of this paper is to show that, given plausible assumptions, the above proof can be reconstructed *without* appeal to contraction or any of its cognates beyond those restricted intances of it which hold for arithmetical rather than logical reasons.

That contraction is the main logical move in all of the usual self-reference paradoxes has been known for at least forty years.[1] Curry produced his brief account of such paradoxes in 1942 [5] and credits Carnap with the central idea. Curry's paradox is a trivialisation of any theory permitting the construction of a predicate C such that

$$\forall F(C(F) \leftrightarrow (F(F) \to B)$$

for arbitrary B. The key instantiation, of course, is

	$C(C) \leftrightarrow (C(C) \to B)$	
$\therefore$	$C(C) \to (C(C) \to B)$	
$\therefore$	$C(C) \to B$	by contraction
$\therefore$	$C(C)$	from this and the biconditional
$\therefore$	B	by detachment.

Geach [7] and Prior in 1955 [9] devised semantic antinomies free of negation and of the concept of falsehood. Both use essentially Curry's derivation and both mention the possibility of contraction-free logic as a way out though neither develops the idea very far. The thought that restriction of contraction may be a viable move was greatly boosted in 1980 when Brady [1, 2][2] announced non-trivial consistency proofs for naïve set theories based on some contractionless systems of logic close to the logic **TW**. It remains to be seen whether an adequate amount of mathematics can be reconstructed without using contraction; it is clear that diagonal arguments such as those of Cantor and many in elementary recursion theory rely heavily on contraction, and it is an open question whether or in what form Gödel's incompleteness theorems can be recovered without it. As van Benthem stresses in his 1978 paper [11], Löb's theorem of 1955 is intimately related to Curry's pardox, and it seems likely that it would fail for contractionless systems of arithmetic. There is a vast amount of formal hard work to be done before we gain a clear vision of the formal sciences without contraction. What appears below scarcely begins that work, and indicates how intricate much of the honest toil is likely to be.

Contraction, then, is profoundly important in the area of foundations, being critically involved in both proofs and paradoxes of the most fruitful kinds. It is also interesting to note that the best known contractionless logics in the literature, the many-valued logics of Łukasaiewicz, represent the most favoured basis for fuzzy logics and for "deviant" approaches to vagueness generally.[3] It would be fascinating to discover through logical investigation an abstract unity

[1]Recall that this passage was written in 1982.

[2]In 1982, these papers were circulating underground in typescript. They were not published until 1989. Later work by Brady [4] extends the result to a range of logics including **TW** itself.

[3]The going program in fuzzy logic seems to me to be confused, mainly because of failure to attend to the philosophical underpinnings of logic and over-readiness to adopt simple-minded arithmetical models of "degrees of truth". I am therefore wary of placing much weight on it; but it may be plundered for some general motivational thoughts.

between the phenomena of self-reference and the traditionally distinct sorites paradox. Fascinating or not, that thought will not be pursued any further in this paper. The roots, formal and historical, of the logics treated here lie not in many-valued logic but in the "relevant" systems of Church, Ackermann, Anderson and Belnap. The guiding motivation for these logics was Church's "use criterion" for validity: in a valid argument, every premise must be used (otherwise it is unavailable for application of the deduction theorem). This is fine, but it emerges on closer inspection that some delicate formal machinery has to be employed in determining what is to count as a use. Consider

$$\cfrac{P \vdash P \qquad\qquad P \vdash P}{\cfrac{P, P \vdash P}{P \vdash P \to P}\ \text{(deduction theorem)}}\ \text{(mingle)}$$

Anderson and Belnap agree with Church that this violates the spirit of the use criterion, and so they arrange things to ensure that it violates the letter also. One has not used *all* of the premises in deriving P from the bunch (P, P). In other words, repetitions count: each premise must be used at least as many times as it is assumed. This invites the opposite question: may each premise be used *at most* as many times as it is assumed? The same authors impose no such restriction, allowing for instance

$$\frac{A \vdash P \qquad\qquad A \vdash P \to Q}{A \vdash Q}\ \text{(detachment)}$$

instead of

$$\frac{A \vdash P \qquad\qquad A \vdash P \to Q}{A, A \vdash Q}\ \text{(detachment)}$$

and generally allowing

$$\frac{X\,A{,}A\,Y \vdash P}{X\,A\,Y \vdash P}\ \text{(contraction)}$$

But clearly there is nothing in the schematic statement of the use criterion which requires us to interpret it in such a lax manner. It is in some ways more faithful to the motivating thoughts of relevant logic to count repetitions strictly, requiring for the purposes of the deduction theorem that the premises be used *exactly* as many times as they are assumed. The result of so doing, naturally, is a contraction-free relevant logic **RW**.

The original goal of the present paper was to prove the irrationality of $\sqrt{2}$ in $\mathbf{RW}^\sharp$, the arithmetic resulting from Meyer's $\mathbf{R}^\sharp$ [8] by weakening the underlying logic from **R** to **RW**. However, **RW** still embodies some strong assumptions in the form of structural rules. For example, it does not distinguish between the pairs of assumptions $A{,}B$ and $B{,}A$, despite centuries of logical tradition in which major premises have been distinguished from minor ones as though it mattered, and despite the fact that applying a function f to another function g is not at all the same as applying g to f. It is therefore interesting and worthwhile to strengthen the result by weakening the logic. Removing all structural rules governing ';' leaves the basic relevant logic **B** to which we now turn.

2 Formulation of a weak arithmetic[4]

We begin with notation. The formal first order language is built up in the normal way from atomic formulas. Since it is a language for arithmetic, there is just one predicate, '=', just one primitive name, '0' and the usual symbols for the successor, addition and multiplication functions. The connective and quantifier symbols

$$\sim \quad \& \quad \vee \quad \rightarrow \quad \forall \quad \exists$$

are as usual. Note that $\rightarrow$ is relevant implication, and is not definable from the rest. The positive fragment of the language has all operators except negation, and the extensional fragment has everything except implication. The positive extensional fragment, which features in the proof below, is the intersection of these two.

At the metalogical level, the upper case letters P, Q, etc. shall be used as free variables over formulae, and lower case a, b, etc. as free variables over terms. $P_{[x \leftarrow a]}$ is the result of substituting term a for all free occurrences of variable x in P. The logic, and by extension the arithmetic, is presented as a system of natural deduction, written in linear fashion, each line representing a sequent. The premises (formulae on the left of the sequent) may be abbreviated to single Greek letters α, β, etc. to simplify notation. A sequent $P \vdash Q$ with just one formula on the left is provable iff the conditional $P \rightarrow Q$ is a theorem. There are two ways in which compound bunches of assumptions may be formed on the left of more complex sequents. The comma ',' is used to symbolise one of these, and the semicolon ';' to symbolise the other. For sequents with two premises and a conclusion, the reading is intuitively clear: $P, Q \vdash R$ is valid (or provable) iff the single-premise sequent $P \,\&\, Q \vdash R$ is, while $P; Q \vdash R$ is valid (or provable) iff $P \vdash Q \rightarrow R$ is. The two pairing operations may be nested to build up bunches of arbitrary complexity. Since the left side of a sequent may not be empty, there is also a special identity bunch $\mathcal{T}$ to indicate theorems.

The upper case letters A, B, etc. are used in the metalanguage to vary over bunches in this technical sense. In specifying the logic we make use of the notation '$X\ A\ Y$' to indicate an arbitrary bunch in which A occurs somewhere as a sub-bunch; then '$X\ B\ Y$' indicates the bunch which is exactly the same except that A is replaced by B. There are structural rules governing the comma: it is associative, commutative and idempotent and satisfies (extensional) weakening in the form

$$\frac{X\ A\ Y \ \vdash\ P}{X\ A, B\ Y \ \vdash\ P}$$

The special bunch $\mathcal{T}$ is a left identity for the semicolon, meaning that $\mathcal{T}; A$ may replace or be replaced by A anywhere.

[4]This section is greatly abridged from the original typescript, which at this point devotes some 9 pages to presenting a natural deduction calculus which was eventually published separately [10] as a "General Logic". There is no reason to reproduce that material here, except to note the minor historical points that it had its roots in Meyer's use of natural deduction for arithmetic [8] and that the word 'bunch' for the premise structures was in use in Canberra as early as 1982. The concept itself has more roots in earlier works by Dunn, Belnap and more recently Giambrone, so it too was familiar to relevant logicians at the time. It was later extended and elaborated by Brady [3] in his work on the proof theory of distributive substructural logics.

The logic **BQ** is the propositional logic **B**, equipped in the standard way[5] with quantifiers. It is defined by means of introduction and elimination rules:

$$\frac{A \vdash P \,\&\, Q}{A \vdash P}\ (\&\text{E}) \qquad\qquad \frac{A \vdash P \,\&\, Q}{A \vdash Q}\ (\&\text{E})$$

$$\frac{A \vdash P \qquad B \vdash Q}{A, B \vdash P \,\&\, Q}\ (\&\text{I})$$

$$\frac{A \vdash P}{A \vdash P \vee Q}\ (\&\text{E}) \qquad\qquad \frac{A \vdash Q}{A \vdash P \vee Q}\ (\&\text{E})$$

$$\frac{X\,P\,Y \vdash R \qquad X\,Q\,Y \vdash R \qquad A \vdash P \vee Q}{X\,A\,Y \vdash R}\ (\vee\text{E})$$

$$\frac{A \vdash P \to Q \qquad B \vdash P}{A; B \vdash Q}\ (\to\text{E}) \qquad\qquad \frac{A; P \vdash Q}{A \vdash P \to Q}\ (\to\text{I})$$

$$\frac{A \vdash \forall x P}{A \vdash P_{[x \leftarrow a]}}\ \begin{array}{l} a \text{ free for } x \text{ in } P \\ (\forall\text{E}) \end{array} \qquad \frac{A \vdash P}{A \vdash \forall x P}\ \begin{array}{l} x \text{ not free in } A \\ (\forall\text{I}) \end{array}$$

$$\frac{A \vdash P_{[x \leftarrow a]}}{A \vdash \exists x P}\ \begin{array}{l} a \text{ free for } x \text{ in } P \\ (\exists\text{I}) \end{array}$$

$$\frac{A \vdash \exists x P \qquad X\,P\,Y \vdash Q}{X\,A\,Y \vdash Q}\ \begin{array}{l} x \text{ not free in } X\_Y \text{ or } Q \\ (\exists\text{E}) \end{array}$$

Thus far everything is much as in familiar logics, including intuitionistic logic but note the comma in &I and the semicolons in →E and →I, which make all the difference. Negation, however, changes the picture a little:

$$\frac{A \vdash {\sim}{\sim}P}{A \vdash P}\ ({\sim}{\sim}\text{E}) \qquad\qquad \frac{A \vdash Q \qquad P \vdash {\sim}Q}{A \vdash {\sim}P}\ ({\sim}\text{IE})$$

This permits, among other things, proofs of the classical duality between the universal and existential quantifiers, whereby the positively derivable sequent

[5]The relevant theory of quantification is not quite as "standard" now as it was in 1982, though note that **BQ** is not subject to Fine's incompleteness result [6] for **RQ** and related logics, so matters are more straightforward than they would be in stronger systems.

$$P \,\&\, \exists x Q \;\vdash\; \exists x(P \,\&\, Q) \qquad x \text{ not free in } P$$

gives rise to the non-constructive confinement principle

$$\forall x(P \vee Q) \;\vdash\; P \vee \forall x Q \qquad x \text{ not free in } P$$

As in the classical case, therefore, the logic given by the positive rules is extended non-conservatively by the addition of negation.

The arithmetic $\mathbf{B}^\sharp$ is a regular **BQ** theory (i.e. it contains as theorems all theorems of **BQ** in its vocabulary) closed under the above rules. Its specifically arithmetical postulates are close to those of the relevant Peano arithmetic $\mathbf{R}^\sharp$:[6]

$$\frac{}{\vdash\; a = a}\;(p1) \qquad \frac{A \;\vdash\; a = b}{A \;\vdash\; b = a}\;(p2)$$

$$\frac{A \;\vdash\; a = b \qquad B \;\vdash\; b = c}{A; B \;\vdash\; a = c}\;(p3)$$

$$\frac{A \;\vdash\; a = b}{A \;\vdash\; a' = b'}\;(p4) \qquad \frac{A \;\vdash\; a' = b'}{A \;\vdash\; a = b}\;(p5)$$

$$\frac{}{\vdash\; \sim(0 = a')}\;(p6)$$

$$\frac{}{\vdash\; a + 0 = a}\;(p7) \qquad \frac{}{\vdash\; a + b' = (a + b)'}\;(p8)$$

$$\frac{}{\vdash\; a \times 0 = 0}\;(p9) \qquad \frac{}{\vdash\; a \times b' = (a \times b) + a}\;(p10)$$

$$\frac{\vdash\; P_{[x \leftarrow 0]} \qquad P \;\vdash\; P_{[x \leftarrow x']}}{\vdash\; P}\;(p11)$$

Note that p11, the induction postulate, is in the form corresponding to a rule rather than to an axiom. In the case of $\mathbf{R}^\sharp$, this makes no difference, but in weak systems such as $\mathbf{B}^\sharp$ it is unclear what stronger form of induction would be well motivated, so here it remains in the rule form.

Note also that p3, the postulate making equality transitive, is in the form corresponding to a nested implication. This appears to be needed for the present proof, but of course it would be possible to weaken that to the rule form (perhaps more in keeping with the **B** view of the world) and to investigate the resulting arithmetic if we so wished.

A key feature of inference in **B** is the principle of *affixing*

$$\frac{P \;\vdash\; Q \qquad R \;\vdash\; S}{Q \to R \;\vdash\; P \to S}$$

[6]For brevity, we follow standard practive in writing '$\vdash\; P$' to mean '$\mathcal{T} \;\vdash\; P$'.

or equivalently of *prefixing* and *suffixing* in the forms:

$$\frac{P \ \vdash \ Q}{R \to P \ \vdash \ R \to Q} \qquad\qquad \frac{P \ \vdash \ Q}{Q \to R \ \vdash \ P \to R}$$

These are easily derivable using the natural deduction rules for implication.

In writing arithmetical formulae in the rest of this paper, we allow ourselves some freedom of notation, writing 'ab' for '$a \times b$', 'a^2' for '$a \times a$', and so forth, expecting that the reader will be familiar with such conventions. We also abbreviate chains of low-level inference steps to single inferences, citing postulates as appropriate and expecting steps of affixing, transitivity of implication, etc. to be understood. The derivable rule

$$\frac{P \ \vdash \ Q_{[x \leftarrow a]}}{\exists x P \ \vdash \ \exists x Q} \quad (\exists\text{IE})$$

(where a is free for x in P) will be used frequently.

3 Proof preliminaries

BQ has many properties one would expect of a logic—albeit a weak one—in the mainstream logical tradition. Conjunction and disjunction satisfy the same distributive lattice conditions as they do in classical or intuitionistic logics. De Morgan's duality principles and the corresponding duality between the quantifiers, are straightforwardly derivable. Implication is governed by the deduction theorem (→I) and its converse. $\mathbf{B}^\sharp$ as well as **BQ** itself is closed under universal generalisation, in the light of ∀I.

As theorems of $\mathbf{B}^\sharp$, we may note the following:

T1. $a + b = b + a$
T2. $a + (b + c) = (a + b) + c$
T3. $ab = ba$
T4. $a(bc) = (ab)c$
T5. $a(b + c) = ab + ac$
T6. $a = b \leftrightarrow a + c = b + c$
T7. $a = b \to (c = d \to a + c = b + d)$
T8. $\exists x(a = b \leftrightarrow 0 = x)$
T9. $a'^2 = a^2 + 2a + 1$ where 1 is $0'$, 2 is $0''$, etc.
T10. $a = b \to (0 = 0 \vee \mathrm{f}^+)$ where f^+ is $0 < 0$
T11. $a < b \to (a = b \to \mathrm{f}^+)$ and $< b$ means $\exists x(a + x' = b)$
T12. $a = 0 \vee \exists x(a = x')$
T13. $0 = 0 \to 0 < a'$

Of particular importance for the present paper is T6, proved by induction on c. The base case, $a = b \leftrightarrow (a + 0 = b + 0)$, follows from p7 in virtue of p2 and p3. The induction step is similarly simple, as $a = b \leftrightarrow (a + x' = b + x')$ is equivalent to $a = b \leftrightarrow ((a + x)' = (b + x)')$ by postulate p8, and the latter is equivalent to $a = b \leftrightarrow (a + x = b + x)$ by p4 and p5, again using p2 and p3 to make the equational inferences.

It is interesting that T1–T13 are provable without using any logical resources beyond those of **B**. They do depend heavily on the induction rule p11, and on the fact that p3, the postulate for the transitivity of identity, holds in the form of a nested implication rather than just as a rule or as an implication with conjoined antecedents. Consider, for example, T7. This follows from T6 in virtue of the rule of suffixing:

1.	$a = b \ \vdash \ a + c = b + c$	T6
2.	$c = d \ \vdash \ b + c = b + d$	T6, T1
3.	$a + c = b + c \ \vdash \ b + c = b + d \rightarrow a + c = b + d$	p3
4.	$a = b \ \vdash \ b + c = b + d \rightarrow a + c = b + d$	1, 3
5.	$b + c = b + d \rightarrow a + c = b + d \ \vdash \ c = d \rightarrow a + c = b + d$	2, suffixing
6.	$a = b \ \vdash \ c = d \rightarrow a + c = b + d$	4, 5

Next, it is useful to note that induction can be generalised, just as classically, to allow proofs by double induction:

LEMMA 1 The rule of double induction

$$\frac{\vdash \ P_{[x\leftarrow 0]} \qquad \vdash \ P_{[y\leftarrow 0]} \qquad P \ \vdash \ P_{[x\leftarrow x',y\leftarrow y']}}{\vdash \ P}$$

is admissible in $\mathbf{B}^\sharp$.

For proof, first observe that the rule of quasi-induction

$$\frac{\vdash \ P_{[x\leftarrow 0]} \qquad \vdash \ P_{[x\leftarrow x']}}{\vdash \ P}$$

is admissible in $\mathbf{B}^\sharp$.

Proof of this is by induction. Suppose the predicate P provably holds of 0 and provably holds of all successors. Then let the predicate Q be $P \,\&\, \forall x P_{[x\leftarrow x']}$. Obviously $\vdash Q_{[x\leftarrow 0]}$ and equally obviously $Q \ \vdash \ Q_{[x\leftarrow x']}$. Hence by ordinary induction, $\vdash Q$ and hence $\vdash P$.

Now the proof that double induction is admissible proceeds as expected by two layers of single induction. Let $P(x, y)$ have free variables x and y and suppose $P(x, 0)$, $P(0, y)$ and $P(x, y) \rightarrow P(x', y')$ (and their universal closures) are all theorems. Let the unary predicate $R(y)$ with free variable y be

$$\forall x((\forall z P(x, z) \,\&\, \forall z P(z, 0)) \rightarrow (P(x', y) \,\&\, \forall z P(z, 0))).$$

Clearly $R(0)$ is a theorem, and equally clearly so is $R(y')$. Hence so is $R(y)$ and by generalisation so is $\forall y R(y)$:

$$\forall y(\forall x((\forall z P(x, z) \,\&\, \forall z P(z, 0)) \rightarrow (P(x', y) \,\&\, \forall z P(z, 0)))).$$

By logical moves such as confinement of the universal quantifier, omitting the leading $\forall x$ and rewriting bound variables:

$$\forall y P(x, y) \,\&\, \forall z P(z, 0) \ \vdash \ \forall y P(x', y) \,\&\, \forall z P(z, 0).$$

Now $\vdash \forall y P(0,y) \,\&\, \forall x P(x,0)$, so by ordinary induction on x, $\vdash P(x,y)$ as required.

One effect of quasi-induction is that the two base cases of the double induction rule may be split into three: the case in which x and y are both zero and the two cases in which one of them is zero and the other is a successor. This simplifies many double induction arguments, and will be used below.

LEMMA 2. $a^2 + a$ is even. That is, $\vdash \forall x \exists y(xx + x = 2y)$.

This is somehow obvious, as $a^2 + a$ is $a(a+1)$ which is the product of two numbers one of which must be even. To prove it carefully from first principles, however, it is as quick to go through a proof by induction:

1.	$0^2 + 0 = 2 \times 0$	trivial
2.	$a'a' = aa + a + a'$	T9
3.	$a'a' + a' = (aa + a) + (a' + a')$	2, T6
4.	$a'a' + a' = (aa + a) + 2a'$	3, T6, etc
5.	$aa + a = 2x \ \vdash \ (aa + a) + 2a' = 2x + 2a'$	T6
6.	$aa + a = 2x \ \vdash \ a'a' + a' = 2(x + a')$	4, 5, T5
7.	$\exists x(aa + a = 2x) \to \exists x(a'a' + a' = 2x)$	6, ∃IE
8.	$\forall x \exists y(xx + x = 2y)$	1, 7, p11

LEMMA 3. If $a + b$ is even, then if a is even, b is even. That is:

$$a + b = 2x \ \vdash \ a = 2y \to \exists z(b = 2z)$$

This is a consequence, by prefixing and the transitivity of implication, of two facts:

L3a. $a + b = 2x;\ a = 2y \ \vdash \ 2x = 2y + b$
L3b. $2x = 2y + b \ \vdash \ \exists z(b = 2z)$

L3a is easily shown:

1.	$a + b = 2x \ \vdash \ (a + b = 2y + b) \to (2x = 2y + b)$	p2, p3
2.	$a = 2y \ \vdash \ a + b = 2y + b$	T6
3.	$a + b = 2x \ \vdash \ a = 2y \to 2x = 2y + b$	1, 2, affixing

L3b is proved by double induction on x and y. The first base case

$$2 \times 0 = 2y + b \ \vdash \ \exists z(b = 2z),$$

is *not* proved by arguing that if $a + b = 0$ then $b = 0$, since this is not relevantly valid arithmetical reasoning: it even fails in $\mathbf{R}^\sharp$. Rather, the argument is that if $0 = 2y + b$ then $b = 2y + 2b$ (adding b to both sides as in T6), and that $2y + 2b = 2(y + b)$ as in T5.

The second base case,

$$2x = 2 \times 0 + b \to \exists z(b = 2z),$$

is simpler, since the antecedent simplifies to $2x = b$ which is all we need.

For the double induction step:

1.	$\alpha \vdash 2x = 2y + b \to \exists z(b = 2z)$		assume
2.	$\beta \vdash 2x' = 2y' + b$		assume
3.	$\beta \vdash 2x + 1 = 2y + b + 1$		2, p9, T6
4.	$\beta \vdash 2x = 2y + b$		3, T2, T6
5.	$\alpha; \beta \vdash \exists z(b = 2z)$		1, 4, $\to$E
6.	$\alpha \vdash (2x' = 2y' + b \to \exists z(b = 2z)$		5 $\to$I

Note again that Lemma 3 does not require prefixing and suffixing to hold in the theorem form, but merely the rule forms. Hence, it too holds in $\mathbf{B}^\sharp$.

A simple corollary which is significant enough to be a named lemma in its own right, is:

LEMMA 4. a^2 is even iff a is even. That is,

$$\vdash \exists x(a^2 = 2x) \leftrightarrow \exists x(a = 2x)$$

Lemma 4 is immediate from lemmas 2 and 3.

LEMMA 5. $2a = 2b \vdash a = b \vee \mathrm{f}^+$

That $2a = 2b$ does not imply $a = b$ even if the underlying logic is strengthened to **RM**, is known from Meyer's work on relevant arithmetic in the early 1970s [8], but the addition of f^+ (i.e. $0 < 0$) as a disjunct suffices to cover the counterexamples. Proof of lemma 5 is again by double induction. The base cases, in which one of a or b is 0, are all trivial, and the induction step is straightforward, given the easy observation that $2x' = 2y'$ is equivalent to $2x = 2y$ in virtue of p10 and T6, while $a' = b'$ is axiomatically equivalent to $a = b$.

LEMMA 6. $\vdash (a = b \to a^2 = b^2) \vee (a = b \to \mathrm{f}^+)$

Proof of this proceeds by way of the stronger:

LEMMA 6a. $\vdash (a = b \to \forall x(a^2 + ax = b^2 + bx)) \vee (a = b \to \mathrm{f}^+)$

Proof is by double induction, and as for lemma 5 the base cases are easy, using the device of splitting the case in which a and b are both 0 from the cases in which one of them is a successor. For the induction case, note that the theorem is of the form $P(a, b) \vee Q(a, b)$, and we need to show that this provably implies $P(a', b') \vee Q(a', b')$. In fact we show that $P(a, b)$ implies $P(a', b')$ and that $Q(a, b)$ implies $Q(a', b')$. The logic of disjunction in **B** does the rest. Note that $a' = b'$ implies $a = b$ by axiom p5, so the induction hypothesis $Q(a, b)$ immediately gives us $Q(a', b')$, while $P(a, b)$ gives

$$a' = b' \to \forall x(aa + ax = bb + bx)$$

so the proof amounts to showing

$$\forall x(aa + ax = bb + bx) \to \forall x(a'a' + a'x = b'b' + b'x)$$

Note that $a'a' = aa + 2a + 1$ (T9), so $a'a' + a'x = aa + a(x + 2) + x + 1$. Similarly, $b'b' + b'x = bb + b(x + 2) + x + 1$. Subtracting $x + 1$ from both sides, $a'a' + a'x = b'b' + b'x$ is provably equivalent to $aa + a(x + 2) = bb + b(x + 2)$, which follows from $\forall x(aa + ax = bb + bx)$ simply by substituting $x + 2$ for x. Proof of the induction step is immediate.

As a corollary to lemma 6, $a = b \vdash a^2 = b^2 \vee \mathrm{f}^+$.

LEMMA 7. $\vdash P \vee (P \to \mathrm{f}^+)$
where the only logical operators in P are $\&$, $\vee$, $\forall$ and $\exists$.

Proof by structural induction on P. The base case, in which P is an equation, is proved by double induction, which is easy given that $a' = b'$ is equivalent to $a = b$. The cases in which P is a conjunction or a disjunction are left to the reader. For the case in which P is of the form $\forall x Q$:

1.	$\vdash Q \vee (Q \to \mathrm{f}^+)$	hypothesis
2.	$\vdash \forall x(Q \vee (Q \to \mathrm{f}^+))$	1, $\forall$-closure
3.	$\vdash \forall x(Q \vee (\forall x Q \to \mathrm{f}^+))$	2, affixing
4.	$\vdash \forall x Q \vee (\forall x Q \to \mathrm{f}^+))$	3, confinement

The case in which P is of the form $\exists x Q$ is similar:

1.	$\vdash Q \vee (Q \to \mathrm{f}^+)$	hypothesis
2.	$\vdash \forall x(Q \vee (Q \to \mathrm{f}^+))$	1, $\forall$-closure
3.	$\vdash \exists x Q \vee \forall x(Q \to \mathrm{f}^+)$	2, affixing, confinement
4.	$\vdash \exists x Q \vee (\exists x Q \to \mathrm{f}^+)$	3, confinement, duality

The quantifier confinement steps at line 4 in the first proof and line 3 in the second are non-constructive but valid in **BQ**.

Postulate p6 is essentially $\sim\mathrm{f}^+$. Hence in strong arithmetics such as $\mathbf{RW}^\sharp$, lemma 7 implies the ordinary law of the excluded middle $P \vee \sim P$ for formulae in the extensional ($\to$ -free) vocabulary. In weaker arithmetics such as $\mathbf{B}^\sharp$, there is no simple argument from lemma 7 to the law of the excluded middle, since $P \to \mathrm{f}^+$ does not provably imply $\sim P$. However, even in such weak logics, $P \vee \sim P$ can be proved for extensional formulae by an argument similar to that of Lemma 7.

The crucial upshot of Lemma 7 is a special case of contraction

$$\frac{P; P \vdash Q}{P \vdash Q}$$

provided P is positive and $\to$ -free, and Q is a consequence of f^+. That is:

LEMMA 8. Where the only operators in P are $\&$, $\vee$, $\forall$ and $\exists$, if $P; P \vdash Q \vee \mathrm{f}^+$ then $P \vdash Q \vee \mathrm{f}^+$

Proof is by simple propositional logic:

1.	$P; P \vdash Q \vee \mathrm{f}^+$	suppose
2.	$P \vdash P \to (Q \vee \mathrm{f}^+)$	1, $\to$I
3.	$\mathrm{f}^+ \vdash Q \vee \mathrm{f}^+$	$\vee$I
4.	$P \to \mathrm{f}^+ \vdash P \to (Q \vee \mathrm{f}^+)$	3, prefixing
5.	$P \vee (P \to \mathrm{f}^+) \vdash P \to (Q \vee \mathrm{f}^+)$	2, 4, $\vee$E
6.	$\mathcal{T} \vdash P \to (Q \vee \mathrm{f}^+)$	5, lemma 7
7.	$P \vdash Q \vee \mathrm{f}^+$	6

It is of interest that lemma 8 does not depend heavily on arithmetical reasoning. Only the case in which P is an equation involves any arithmetic, to establish the base case for the structural induction in the proof of lemma 7. That base case is proved by a double induction, requiring only the trivial p1 and the equivalence constituted by p4 and p5. The rest is just logic, and since it goes through in **BQ**, not much of that. Hence, in a *very* weak arithmetic, a restricted form of contraction is admissible—restricted, but enough to secure the goal of the present paper, as we shall see.

LEMMA 9. (i) $\mathrm{f}^+ \vdash a = b \to \mathrm{f}^+$, and (ii) $a = b \vdash \mathrm{f}^+ \to \mathrm{f}^+$.

(i) is proved simply:

1.	$0 = x' \vdash y = (x + y)'$	p7, p8, T6
2.	$0 = x' \vdash 0 = y \to 0 = (x + y)'$	1, p2, p3
3.	$\mathrm{f}^+ \vdash 0 = y \to \mathrm{f}^+$	2, ∃IE, definition
4.	$\mathrm{f}^+ \vdash \exists y(0 = y) \to \mathrm{f}^+$	3, ∃E
5.	$a = b \vdash \exists y(0 = y)$	from T8
6.	$\mathrm{f}^+ \vdash a = b \to \mathrm{f}^+$	4, 5, affixing

(ii) is equivalent to (i) in $\mathbf{RW}^\sharp$ just by permuting the antecedents, but in $\mathbf{B}^\sharp$ it calls for a separate proof:

1.	$\vdash \exists y(a = b \leftrightarrow 0 = y)$	T8
2.	$0 = y \vdash 0 + x' \to y + x'$	T6
3.	$0 = y \vdash 0 = 0 + x' \to 0 = y + x'$	2, p3
4.	$\vdash x' = 0 + x'$	p2, p7
5.	$0 = x' \vdash 0 = 0 + x'$	4, T6
6.	$0 = 0 + x' \to 0 = y + x' \vdash 0 = x' \to 0 = y + x'$	5, affixing
7.	$0 = y \vdash 0 = x' \to 0 = (y + x)'$	3, 6, p8
8.	$0 = y \vdash \mathrm{f}^+ \to \mathrm{f}^+$	7, ∃IE
9.	$a = b \vdash \mathrm{f}^+ \to \mathrm{f}^+$	1, 8

The two theorems given as lemma 9 are special cases of paradoxes of implication, but they hold in $\mathbf{B}^\sharp$ for arithmetical rather than purely logical reasons. In $\mathbf{R}^\sharp$ they hold in a stronger form, with any formula P (in the $\mathbf{R}^\sharp$ vocabulary) in place of f^+.

LEMMA 10. $2a'^2 = b'^2 \vdash a < b \vee \mathrm{f}^+$

As in the case of lemma 6, we prove the stronger result

$$\vdash \forall x \forall y(2a'^2 + xa' + y = b'^2 + xb' \to a < b \vee \mathrm{f}^+)$$

Recall that $a < b$ is defined to mean $\exists z(a + z' = b)$. Now proof is by double induction on a and b. For the base cases, first if $b = 0$ then $b'^2 + xb'$ is just $x + 1$. But

1.	$2a'^2 = 2a^2 + 4a + 2$	T9, T5
2.	$2a'^2 + xa' + y = 2a^2 + 4a + 2 + xa + x + y$	1, T6, p8

3.	$2a'^2 + xa' + y = (2a^2 + 4a + xa + y)' + (x + 1)$	2, T2, etc
4.	$2a'^2 + xa' + y = x + 1 \to (2a^2 + 4a + xa + y)' = 0$	3, T6
5.	$2a'^2 + xa' + y = b'^2 + xb' \to \mathrm{f}^+$	4, ∃I , def

Secondly, assuming that $a = 0$ and b is a successor c':

1.	$2a'^2 + xa' + y = b'^2 + xb' \to (0 = 0 \vee \mathrm{f}^+)$	T10
2.	$0 = 0 \to 0 < c'$	T13
3.	$(0 = 0 \vee \mathrm{f}^+) \to (a < b \vee \mathrm{f}^+)$	2, ∨I, ∨E

Now, for the induction step, recall that we are working with an antecedent $P(a, b)$ defined as $2a'^2 + xa' + y = b'^2 + xb'$. For induction, we need to show that $\forall x(P(a, b) \to (a < b \vee \mathrm{f}^+))$ implies $\forall x(P(a', b') \to (a' < b' \vee \mathrm{f}^+))$. Clearly, $a' < b'$ is equivalent to $a < b$, so the two consequents may be regarded as identical. Abbreviating $a < b \vee \mathrm{f}^+$ to Q, the problem anounts to proving

$$\forall x(P(a, b) \to Q) \to \forall x(P(a', b') \to Q)$$

The argument proceeds by showing that every instance of $P(a', b')$ provably implies an instance of $P(a, b)$. Suffixing and the **B** logic of quantifiers will then secure the result. So start by assuming an arbitrary instance of $P(a', b')$:

1.	$\alpha \vdash 2a''^2 + xa'' + y = b''^2 + xb''$	assume
2.	$\alpha \vdash 2a'^2 + 4a' + 2 + xa' + y + x = b'^2 + 2b' + 1 + xb' + x$	1, T9
3.	$\alpha \vdash 2a'^2 + 4a' + 1 + xa' + y = b'^2 + 2b' + xb'$	2, T6
4.	$\alpha \vdash 2a'^2 + x''a' + (2a' + y)' = b'^2 + x''b'$	3, T2, etc

Formula 4 is an instance of $P(a, b)$ as required.

LEMMA 11. $a < b' \vdash a = b \vee a < b$

Proof is easy: if $a < b'$ then (by definition) $a + x' = b'$ for some x. It follows by p8 that $(a + x)' = b'$ and so by p5, $a + x = b$. By T12, it is a theorem that $x = 0 \vee \exists y(x = y')$; but if $x = 0$ then $a = b$, while if $\exists y(x = y')$ then $a < b$. Expressing this derivation as a formal proof in $\mathbf{B}^\sharp$ is straightforward.

4 The Proof

Armed with these lemmas, let us return to the main proof. The next step is to define two predicates, G and H:

$$G(x) =_{\mathrm{df}} \exists z(x'^2 = 2z'^2)$$
$$H(x) =_{\mathrm{df}} \exists y(y < x \,\&\, G(y))$$

The core of the traditional Greek proof is to establish

$$G(a) \vdash H(a)$$

which is relevantly invalid. What we have instead is the

GREEK LEMMA I. $G(a) \vdash H(a) \vee \mathrm{f}^+$

Proof: We take the proof in stages. First note that by the elementary logic of the existential quantifier it suffices that we prove

G1 $a'^2 = 2b'^2 \ \vdash \ (a > b \ \& \ \exists x(b'^2 = 2x'^2)) \vee \mathrm{f}^+$

which comes by the distribution of & over $\vee$ from the two facts:

G1a $a'^2 = 2b'^2 \ \vdash \ a > b \vee \mathrm{f}^+$
G1b $a'^2 = 2b'^2 \ \vdash \ \exists y(b'^2 = 2y'^2) \vee \mathrm{f}^+$

G1a is lemma 10, so it remains only to prove G1b. We argue:

1	$a' = 2c' \ \vdash \ a'^2 = 4c'^2 \vee \mathrm{f}^+$	lemma 6
2	$\alpha \ \vdash \ a'^2 = 2b'^2$	assume
3	$\alpha \ \vdash \ a'^2 = 4c'^2 \to 2b'^2 = 4c'^2$	p2, p3
4	$\alpha \ \vdash \ \mathrm{f}^+ \to \mathrm{f}^+$	lemma 9
5	$\alpha \ \vdash \ (a'^2 = 4c'^2 \vee \mathrm{f}^+) \to (2b'^2 = 4c'^2 \vee \mathrm{f}^+)$	3, 4, $\vee$I, $\vee$E
6	$\alpha \ \vdash \ a' = 2c' \to (2b'^2 = 4c'^2 \vee \mathrm{f}^+)$	1, 5, affixing
7	$\alpha \ \vdash \ \exists y(a' = 2y') \to \exists y(2b'^2 = 4y'^2 \vee \mathrm{f}^+)$	6, $\exists$IE
8	$\alpha \ \vdash \ \exists y(a' = 2y')$	lemma 4
9	$\alpha;\alpha \ \vdash \ \exists y(2b'^2 = 4y'^2 \vee \mathrm{f}^+)$	7, 8, $\to$E
10	$\alpha;\alpha \ \vdash \ \exists y(2b'^2 = 4y'^2) \vee \mathrm{f}^+$	9, $\vee$E, $\exists$E

Here we need in effect to apply contraction to line 10; contraction is not generally available, but lemma 8 comes to the rescue:

11	$\alpha \ \vdash \ \exists y(2b'^2 = 4y'^2) \vee \mathrm{f}^+$	10, lemma 8
12	$2b'^2 = 4c'^2 \ \vdash \ b'^2 = 2c'^2 \vee \mathrm{f}^+$	lemma 5
13	$\exists y(2b'^2 = 4y'^2) \ \vdash \ \exists y(b'^2 = 2y'^2) \vee \mathrm{f}^+$	12, $\exists$IE
14	$\alpha \ \vdash \ \exists y(b'^2 = 2y'^2) \vee \mathrm{f}^+$	11, 13

That is the heart of the proof. The next step is to extend it a little:

GREEK LEMMA II. $\quad H(a') \ \vdash \ H(a) \vee \mathrm{f}^+$

This is necessary to the proof by induction. At first sight, it may appear that the first Greek lemma provides all we need to secure the theorem by course of values induction, but it is not so. In weaker substructural logics, attempts to establish a properly formed induction step founder on the fact that $a = b \ \& \ P(a)$ does not imply $P(b)$. The induction lemma is proved using the same technique as above to emulate the required contraction. We need to show that $\exists x(x < a' \ \& \ G(x))$ implies $\exists x(x < a \ \& \ G(x)) \vee \mathrm{f}^+$.

1.	$\alpha \ \vdash \ b < a' \ \& \ G(b)$	assume
2.	$\alpha \ \vdash \ (a = b \ \& \ G(b)) \vee (b < a \ \& \ Gb)$	1, lemma 11
3.	$\beta \ \vdash \ a = b \ \& \ G(b)$	assume
4.	$\gamma \ \vdash \ a = b \to a^2 = b^2$	assume
5.	$\gamma;\beta \ \vdash \ a^2 = b^2$	3, 4
6.	$\gamma;\beta \ \vdash \ b^2 = 2z^2 \to a^2 = 2z^2$	5, p2, p3
7.	$\gamma;\beta \ \vdash \ G(b) \to G(a)$	6, $\exists$IE
8.	$\gamma;\beta \ \vdash \ G(b) \to (G(a) \vee \mathrm{f}^+)$	7
9.	$\delta \ \vdash \ a = b \to \mathrm{f}^+$	assume

10.	$\delta \ \vdash \ a = b \to (G(b) \to \mathrm{f}^+)$	9, lemma 9
11.	$\delta;\beta \ \vdash \ G(b) \to \mathrm{f}^+$	3, 10
12.	$\delta;\beta \ \vdash \ G(b) \to (G(a) \vee \mathrm{f}^+)$	11
13.	$\beta \ \vdash \ G(b) \to (G(a) \vee \mathrm{f}^+)$	8, 12, lemma 6, $\vee$E
14.	$\beta \ \vdash \ G(b) \to (H(a) \vee \mathrm{f}^+)$	13, Greek lemma I
15.	$\beta;\beta \ \vdash \ H(a) \vee \mathrm{f}^+$	3, 14, &E, $\to$E
16.	$\beta \ \vdash \ H(a) \vee \mathrm{f}^+$	15, lemma 8
17.	$\epsilon \ \vdash \ b < a \ \& \ G(b)$	assume
18.	$\epsilon \ \vdash \ H(a) \vee \mathrm{f}^+$	17, $\exists$I , $\vee$I
19.	$\alpha \ \vdash \ H(a) \vee \mathrm{f}^+$	1, 16, 18, $\vee$E
20.	$H(a') \ \vdash \ H(a) \vee \mathrm{f}^+$	19, $\exists$E

There remains only the

THEOREM. $\forall x(G(x) \to \mathrm{f}^+)$

Proof:

1	$\vdash \ (H(0) \vee \mathrm{f}^+) \to \mathrm{f}^+$	easy
2	$(H(x) \vee \mathrm{f}^+) \to \mathrm{f}^+ \ \vdash \ (H(x') \vee \mathrm{f}^+) \to \mathrm{f}^+$	Greek lemma II
3	$\vdash \ (H(x) \vee \mathrm{f}^+) \to \mathrm{f}^+$	1, 2, induction, $\forall$I
4	$\vdash \ G(x) \to \mathrm{f}^+$	3, Greek lemma I
5	$\vdash \ \forall x(G(x) \to \mathrm{f}^+)$	4, $\forall$I

So the assumption that the equation $a^2 = 2b^2$ has a solution in the positive integers is shown to entail, even in the very weak arithmetic $\mathbf{B}^\sharp$, the standard falsehood that 0 is a successor. The proof is carried out entirely in the positive (negation-free) fragment of the theory. Since the negative axiom of Peano arithmntic says exctly that 0 is *not* a successor, we may obtain in one step

$$\vdash \ a'^2 \neq 2b'^2$$

5 Remarks[7]

The purpose of the foregoing was not to convince anyone of the irrationality of $\sqrt{2}$. Nor was it really to show that the theorem can be established in very weak theories of arithmetic: after all, more modern proofs of the same result exist and appear to be logically benign.[8] Nor was it to argue that $\mathbf{B}^\sharp$ in the form presented here is the One True Arithmetic. Rather, it was to reconstruct something very close to the traditional Pythagorean proof in a relevant arithmetic where the underlying logic is free of contraction, thus showing how to recover some important cases of contraction by means of provable disjunctions. Indeed, as the proof goes through in arithmetic based on **B**, it seems that the structural rules

[7]Added in 2024.

[8]For example, let $p(x)$ be the number of prime factors of x. If x is prime, $p(x) = 1$. In any case $p(xy) = p(x) + p(y)$, and $p(x^y) = y.p(x)$. Trivially, if $a^k = nb^k$ then $p(n)$ is a multiple of k. So for any $k > 1$, the k-th root of any prime is irrational. A similar counting argument quickly yields the stronger result that for any positive n, if $a^k = n.b^k$ where a and b are relatively prime, then $b = 1$.

characterising more familiar systems are, in fact, contributing almost nothing to such arithmetical proofs.

The arithmetical postulates used here have some features which introduce styles of reasoning not native to such a meagre logic as **B**. In particular, the transitivity postulate p3 is more in the style of **TW** than of weaker systems. Once again, it is included neither out of homage to Meyer's $\mathbf{R}^\sharp$, nor because I particularly wish to endorse it, but because it is part of the apparatus making the present proof possible.

The interest of this project, then, is in the journey rather than the destination. How weak can our logic get and still be usable as a basis for traditional mathematical reasoning? That question is perhaps too wide to admit of a neat answer, but the present proof may at least provide us with a landmark and an indicator of some of the features of theories in its vicinity. The exercise of proving even the simplest theorems in a theory like $\mathbf{B}^\sharp$ is interesting, to say the least: sometimes fascinating; more often frustrating or even painful. Just what *does* it take to show that if $a^2 < b^2$ then $a < b$? Is it even true, if we let our world view be seriously paraconsistent? Can we show that whatever is true of a and of every number smaller than a is true of all numbers less than the successor of a? This is mathematics under a microscope: the questions induce a kind of mental cramp,[9] as we come to feel that *nothing* in arithmetic can be taken for granted. Hilbert famously complained that denying mathematicians the law of the excluded middle was like denying a boxer the use of his fists. In $\mathbf{B}^\sharp$, not only are we denied all the structural rules whose lack makes logic substructural, but even equational reasoning is limited by the fact that $a = b$ does not imply $ac = bc$, so that under assumptions, equality need not even be a congruence. The boxer has to fight blindfolded without leaving his chair. The surprising fact is that he can, in fact, fight—that theorems can still be proved, and that even the manner of their proof can be emulated quite closely, at least in some non-trivial cases.

One feature, at least, deserves comment. Paraconsistent theories throw the emphasis on their positive fragments. In arithmetic, it seems that $P \to \mathrm{f}^+$ or something similar is the function of P which does the real work; paraconsistent negation contributes very little. In a strong relevant arithmetic such as $\mathbf{R}^\sharp$, it may do more, but in proofs like the present one in $\mathbf{B}^\sharp$, apart from allowing the non-constructive positive inferences to be captured by natural deduction, it literally does not appear until right at the end when the theorem of the form $\sim P$ is inferred from the corresponding $P \to \mathrm{f}^+$. In classical logic, where reasoning is primarily driven by the need to avoid contradictions, negation occupies a central position. The move to paraconsistent logic removes it from this position, for which reason we need a *sui generis* implication connective to form the "heart of logic". By the time we reach **B** as a basis for reasoning, negation is banished to the perimiter, and most of mathematics proceeds without it.[10]

[9] The echo of Wittgenstein is deliberate. In arithmetic, however, we *do* make progress by analysis, but in painfully small steps, by elucidating the fine detail of proofs.

[10] Thanks are due to members of the Australian National University logic group, especially Bob Meyer, Richard Sylvan and Steve Giambrone, for discussions following the first presentation of this result, and to the anonymous reviewer of the current paper whose comments on the present version led to important clarifications.

References

[1] Ross T. Brady. The non-triviality of dialectical set theory. In G. Priest, R. Routley, and J. Norman, editors, *Paraconsistent Logic, Essays on the Inconsistent*, pages 437–471. Philosoiphia Verlag, Munich, 1989.

[2] Ross T. Brady. The non-triviality of extensional dialectical set theory. In G. Priest, R. Routley, and J. Norman, editors, *Paraconsistent Logic, Essays on the Inconsistent*, pages 415–436. Philosoiphia Verlag, Munich, 1989.

[3] Ross T. Brady. Gentzenizations of relevant logics with distribution. *Journal of Symbolic Logic*, 61:402–420, 1996. DOI: 10.2307/2275666.

[4] Ross T. Brady. The simple consistency of naive set theory using metavaluations. *Journal of Philosophical Logic*, 43:261–281, 2014. DOI: 10.1007/s10992-012-9262-2.

[5] Haskell B. Curry. On the inconsistency of certain formal logics. *Journal of Symbolic Logic*, 7:115–117, 1942. DOI: 10.2307/2269292.

[6] Kit Fine. Incompleteness for quantified relevance logics. In J. Norman and R. Sylvan, editors, *Directions in Relevant Logic*, pages 205–225. Springer, Dordrecht, 1989. DOI: 10.1007/978-94-009-1005-8_16.

[7] Peter T. Geach. On insolubilia. *Analysis*, 15:71–72, 1955. DOI: 10.1093/analys/15.3.71.

[8] Robert K. Meyer. Arithmetic formulated relevantly. *Australasian Journal of Logic*, 18:154–288, 2021. DOI: 10.26686/ajl.v18i5.6905.

[9] Arthur Prior. Curry's paradox and 3-valued logic. *Australasian Journal of Philosophy*, 33:177–182, 1955. DOI: 10.1080/00048405585200201.

[10] John Slaney. A general logic. *Australasian Journal of Philosophy*, 68:74–88, 1990. DOI: 10.1080/00048409012340183.

[11] Johann F. A. K. van Benthem. Four paradoxes. *Journal of Philosophical Logic*, 7:49–72, 1978. DOI: 10.1007/bf00245920.