A 2 Set-up Ternary Relational Semantics for the Companions to Brady's 4-valued logic BN4

Sandra M. López*

Abstract

The present paper is inspired by Ross T. Brady's work on semantics for many-valued logics which also belong to the family of relevance. In particular, we aim to enhance his methodology for (meta)completeness results with respect to a 2 set-up ternary relational semantics. In 1982, Brady developed the 4-valued logic BN4 and endowed it with such semantics, providing both strong soundness and completeness theorems. In the recent literature, six new 4-valued logics have been defined as companions to the system BN4 and endowed with a bivalent Belnap-Dunn type semantics. The aim of this paper is to deepen the knowledge of these new companions to BN4 by providing a 2 set-up ternary relational semantics, thus following the same strategies Brady applied to BN4 in [6].

Keywords: Relevant logics; 4-valued logics; Brady's logic BN4; many-valued logics; 2 set-up semantics.

1 Introduction

The significance of Ross T. Brady's work regarding relevant logics (and more precisely, weak relevant logics) is by now well-known (see, for instance, [5, 7]). The present paper is inspired by his research regarding logics from the relevance family; in particular, it is inspired by both his methodology for (meta)completeness results and the logics he studied in [6]. In that article, Brady focused on developing two new interesting many-valued logics within the family of relevance: the three-valued logic RM3 and the four-valued logic BN4. He also endowed those couple of logics with a 2 set-up Routley-Meyer relational semantics and proved that both logics –RM3 and BN4 –are sound and complete with respect to them.

The aim of this paper is twofold: (1) to deepen the knowledge of some 4-valued logics recently developed as companions to Brady's system BN4 [10]; (2) to follow Brady's completeness methodology as applied to BN4 in [6] and provide these new logics with a 2 set-up Routley-Meyer semantics.

In the following lines, we aim to summarise both the importance of Brady's work regarding BN4 and the methods he employed in [6] as well as the main reasons why the present paper contributes to his research on this particular topic and, from a more general perspective, expands the knowledge on (weak) relevant logics [1] and their characteristic semantics. Firstly, we begin by enhancing the role of Brady's logic BN4 in the intersection between relevant and many-valued logics.

Brady developed the logic BN4 in 1982 by taking as the starting point the axiomatization of the basic system B of Routley *et al.* ([22], Chapter 4). However, the system BN4 can also be seen as an implicative expansion of Belnap's logic B4 –which is equivalent to Anderson and Belnap's well-known system FDE (cf. [1, 2, 13]). As a matter of fact, the system BN4 has the same characteristic 4-valued matrix set as Belnap's B4, one of the values being 'n', representing neither truth nor falsity. According to Brady, the two factors just mentioned motivated the label BN4 (cf. [6], p. 32, note 1). On the other hand, the importance of this system was summarised by Routley *et al.* as follows: "BN4 is the correct logic for the 4-valued situation where the extra values are to be interpreted in the both and neither senses" ([11], p. 253). Additionally, this system has even been regarded by some as the adequate extension of FDE, were the latest to be expanded by means of a relevant conditional akin to that of the relevant logic R [23].

*Universidad de Valladolid (Spain) – sandralv@uva.es – ORCID: 0000-0003-2584-5950

[1] At least on some quasi-relevant logics weaker than R and E, although stronger than some canonically weak logics such as Routley and Meyer's logic B.

Over the last decade, Robles and Méndez conducted some research concerning the logic E4 –another interesting 4-valued logic which was built upon a modification of the conditional function characteristic of BN4. This new 4-valued logic was developed by them as a companion to BN4 worthy of consideration among relevant and many-valued logics [17]. As a matter of fact, both of these systems, BN4 and E4, can be respectively considered as the 4-valued logics of the relevant conditional and (relevant) entailment, according to Robles and Méndez.

In the conclusions of [17], Robles and Méndez suggest that there might be other interesting companions to BN4 and propose six different alternatives to the conditional functions characteristic of E4 and BN4. Research on the logics built upon those alternative (implicative) tables has been recently carried out in [10]. The set formed by the aforementioned logics –labelled throughout this paper as L*ti*-logics [2] –can be presented as the class of all implicative expansions of Belnap's logic B4 containing Routley and Meyer's logic B while maintaining the conditional structure of MBN4 or ME4 (i.e., the matrices upon which BN4 and E4 were initially built). Regarding their interest, it is worth underlining that all the logics included in the said class are 4-valued paraconsistent and paracomplete logics with natural conditionals –in the sense of Tomova (cf. [24])–that enjoy some of the characteristic properties of relevant logics (as it is proven in [10], Section 9).

We now proceed to briefly sum up the origins and clarify the place of ternary relational semantics (in particular, 2 set-up models) among non-classical logics.

In the early 70s of the last century, ternary relational semantics (also generally known as Routley-Meyer relational semantics) was introduced (cf. [21, 20, 19]) in order to characterise relevant logics and deal with some problems regarding their metalogical properties [1]. However, it was soon noticed that ternary relational semantics was a useful tool for modelling different types of logics (see, for instance, [8, 18, 16]).

On the other hand, the 2 set-up models (of Routley-Meyer semantics) are based upon a restriction of the virtually infinite elements of the general models to just two. The origin of this particular kind of models in ternary relational semantics seems however quite unclear and, although Brady's paper leads the attention towards [22], a precedent of such models may be found in [12] [3].

In [6], Brady also developed a 2 set-up Routley-Meyer relational semantics for the logics RM3 and BN4. Concerning his strategy, he remarked: "the method of proof is fairly general and it is hoped that other model structures can be axiomatized by appropriate modifications to the proof" ([6], p. 9). Then, following the method developed by Brady as applied to the logic E4 in [17], we shall provide the L*ti*-logics with such semantics. The present paper also shows that these logics are indeed strongly sound and complete with respect to this semantics.

While general Routley-Meyer semantics has captured a great deal of attention since its inception, there have been in comparison very few works regarding the 2 set-up models [4]. Thus, the proposed research will not only deepen our knowledge of those new systems –the so called L*ti*-logics–within the family of (quasi-)relevant logics but also contributes to expand the range of application of both 2 set-up relational semantics and Brady's strategies for completeness results (as applied by him to other systems of the said family).

The paper is divided into five sections. In section 2, all the systems considered in this paper (i.e., the L*ti*-logics) are introduced together with the matrices upon which they were initially built. Then, a 2 set-up Routley-Meyer semantics is developed for them and a proof of the soundness theorem is provided in section 3. In section 4, we prove a series of necessary definitions and preliminary lemmas of use in Section 5, where strong completeness of the L*ti*-logics is finally proved.

2 Systems considered in this paper: the L*ti*-logics

In this section, we display the systems of interest in this paper –the L*ti*-logics. First, we show the matrices upon which the logics considered in this paper were built, i.e., the implicative variants of

[2]Where i refers to a numerical value assigned to each one of these logics. In particular, we are presented with eight different systems: Lt1 is the label used for BN4 and Lt5 for E4. Regarding the other six, only Lt2 has been independently studied and given a different specific label –i.e., EF4 [3].

[3]The reader is advised to consult the introduction of [4] for a more detailed picture of the origins of this specific kind of models in Routley-Meyer semantics.

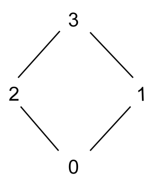
[4]Appart from the precedents mentioned above and Brady's own work [6], we may direct any interested reader to [15], where the first companion to BN4 –the logic E4–was endowed with a 2 set-up Routley-Meyer semantics, and to [4], where the relation between reduced general models and 2 set-up models in ternary relational semantics is displayed.

MBN4 and ME4 which verify Routley and Meyer's logic B (cf. [10]).

The notions of Languages and Logics are fairly standard. The propositional language $\mathcal{L}$ consists of a denumerable set of propositional variables p_0, p_1..., p_n, ... and some or all of the following connectives $\rightarrow, \wedge, \vee, \neg$. A, B, C, etc. are metalinguistic variables. Logics are formulated as Hilbert-style axiomatic systems. The notions of *proof* and *theorem* are understood as it is customary ($\Gamma \vdash_L A$ means that A is derivable from the set of wffs Γ in the logic L; and $\vdash_L A$ means that A is a theorem of the logic L).

Now, the matrices which determine the systems of interest for this paper are shown.

Definition 2.1 (The matrices which determine the Lt*i*-logics –Mt*i*) The propositional language $\mathcal{L}$ consists of the connectives $\rightarrow, \wedge, \vee$ and $\neg$. The matrices Mti are the structures $< \mathcal{V}, \mathcal{D}, F >$, where (i) $\mathcal{V}$ is {0, 1, 2, 3} and it is partially ordered as shown in the following lattice:



(ii) $\mathcal{D} = \{2, 3\}$; (iii) $F = \{f_\rightarrow, f_\wedge, f_\vee, f_\neg\}$ where $f_\wedge$ and $f_\vee$ are defined as the glb (or lattice meet) and the lub (or lattice join), respectively. $f_\neg$ is an involution with $f_\neg(0) = 3$, $f_\neg(3) = 0$, $f_\neg(1) = 1$ and $f_\neg(2) = 2$. Tables for $\wedge$, $\vee$ and $\neg$ are now displayed.

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

	0	1	2	3
$\neg$	3	1	2	0

Finally, $f_\rightarrow$ is defined in each system according to the following tables[5]:

t1 (BN4)

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	1	3	1	3
2	0	1	2	3
3	0	1	0	3

t5 (E4)

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	2	0	3
2	0	0	2	3
3	0	0	0	3

t2

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	3	0	3
2	0	0	2	3
3	0	0	0	3

t3

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	1	3	1	3
2	0	0	2	3
3	0	0	0	3

t4

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	3	0	3
2	0	1	2	3
3	0	1	0	3

t6

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	2	0	3
2	0	1	2	3
3	0	0	0	3

t7

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	2	1	3
2	0	0	2	3
3	0	0	0	3

t8

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	2	1	3
2	0	1	2	3
3	0	0	0	3

Remark 2.2 (Implicative variants of MBN4 and ME4 which verify Routley and Meyer's logic B) The matrices considered in this paper are the only implicative variants of MBN4 (t2-t4) and ME4 (t6-t8) which verify Routley and Meyer's logic B (cf. [22, Chapter 4]). This fact was already proved as a Proposition in [10, Proposition 3.2].

[5] From now on, the labels t1 and t5 will be used to refer to the implicative tables of MBN4 and ME4, respectively. Labels t2-t4 will be used to refer to the implicative tables of the variants of MBN4 and, likewise, t6-t8 will be used for those of the implicative variants of ME4.

The eight logics considered in this paper are developed in this section as extensions of b4, the system shown below (cf. Definitions 2.3 and 2.6). Therefore, the logic b4 is a system contained in every Lti-logic ($1 \leq i \leq 8$), i.e., in every logic built upon the matrices characterized by the implicative tables displayed in Definition 2.1. As a matter of fact, the label b4 is intended to abbreviate "basic logic contained in every companion of BN4 or E4 which includes Routley and Meyer's logic B". We call the logic b4 basic in the sense that it has a mere instrumental role: it serves as a common ground to build all the Lti-logics.

Definition 2.3 (The basic logic b4) The logic b4 is axiomatized with the following axioms and rules ADJ, MP, dMP, dPREF, dSUF, dCON, dCTE displayed below:
Axioms

A1. $A \to A$
A2. $(A \wedge B) \to A$ / $(A \wedge B) \to B$
A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
A4. $A \to (A \vee B)$ / $B \to (A \vee B)$
A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$
A7. $\neg\neg A \to A$
A8. $A \to \neg\neg A$
A9. $\neg A \to [A \vee (A \to B)]$
A10. $B \to [\neg B \vee (A \to B)]$
A11. $(A \vee \neg B) \vee (A \to B)$
A12. $(A \to B) \vee [(\neg A \wedge B) \to (A \to B)]$

Rules of inference

Adjunction: $A, B \Rightarrow A \wedge B$
Modus Ponens: $A, A \to B \Rightarrow B$
Disjunctive Modus Ponens: $C \vee A, C \vee (A \to B) \Rightarrow C \vee B$
Disjunctive Prefixing: $C \vee (A \to B) \Rightarrow C \vee [(D \to A) \to (D \to B)]$
Disjunctive Suffixing: $C \vee (A \to B) \Rightarrow C \vee [(B \to D) \to (A \to D)]$
Disjunctive Contraposition: $C \vee (A \to B) \Rightarrow C \vee (\neg B \to \neg A)$
Disjunctive Counterexample: $C \vee (A \wedge \neg B) \Rightarrow C \vee \neg(A \to B)$

Remark 2.4 (About the instrumental system b4) b4 is the result of adding the axioms A9-A12 and rules dMP, dPREF, dSUF, dCON and dCTE to Routley and Meyer's basic logic B (cf. [22, Chapter 4]). As a matter of fact, b4 can be seen as an extension of dB (i.e., the disjunctive version of Routley and Meyer's logic B). In particular, non-disjunctive rules PREF, SUF, CON and CTE can easily be derived from their disjunctive version plus the rule MP, A4 and T1 (cf. Definition 2.5).

Next, I prove some theorems of b4 which will be useful throughout this paper. Since b4 is contained in all the Lti-logics, these formulae are also theorems of the Lti-logics.

Proposition 2.5 (Some theorems of b4) The following theorems are derivable in b4.

T1 $A \leftrightarrow (A \vee A)$
T2 $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
T3 $(\neg A \wedge \neg B) \leftrightarrow \neg(A \vee B)$

Proof. T1-T3 are theorems of B (actually, they are theorems of FDE; cf. [1, p. 158]), a system included in b4. ■

In the following lines, I introduce the extensions of b4 which I have referred to from the beginning of the section. First of all, I define the notion of extensions (and expansions) of a propositional logic.

Definition 2.6 (Extensions and expansions of a propositional logic L) Let L be a logic formulated with axioms a_1,..., a_n and rules of derivation r_1,..., r_m. A logic L' includes L iff a_1,..., a_n are theorems of L' and rules r_1,..., r_m are provable in L'. If such were the case, L' would be either an extension of L (i.e., a strengthening of L in the language of L) or an expansion of it (i.e., a strengthening of

L in an expansion of the language of L). We shall generally refer to extensions of a logic L by EL-logics.

Given the previous definition, it is clear that the Lti-logics we introduce in Definition 2.7 are Eb4-logics, i.e., extensions of the basic logic b4.

Definition 2.7 (The Lti-logics) By Lti-logics ($1 \leq i \leq 8$), we refer to the logics built upon the matrices Mti ($1 \leq i \leq 8$) shown in Definition 2.1. Each Lti-logic is the result of adding the following axioms (from the list below) to b4:

Lt1 (BN4): A13-A15
Lt2: A16-A22
Lt3: A13, A14, A17, A18, A21-A23
Lt4: A15, A16, A19-A21
Lt5 (E4): A16-A20, A22, A24-A26
Lt6: A16, A21, A22, A23, A24, A27, A28
Lt7: A13, A17, A18, A20, A22, A25, A29
Lt8: A13, A20, A22, A25, A28, A29

Now, I display the list of axioms from which the Lti-logics are built:

A13. $(A \wedge \neg B) \rightarrow [(A \wedge \neg B) \rightarrow \neg(A \rightarrow B)]$

A14. $A \vee [\neg(A \rightarrow B) \rightarrow A]$

A15. $\neg B \vee [\neg(A \rightarrow B) \rightarrow \neg B]$

A16. $[A \wedge (A \rightarrow B)] \rightarrow B$

A17. $[(A \rightarrow B) \wedge \neg B] \rightarrow \neg A$

A18. $A \rightarrow [B \vee \neg(A \rightarrow B)]$

A19. $\neg B \rightarrow [\neg A \vee \neg(A \rightarrow B)]$

A20. $[\neg(A \rightarrow B) \wedge \neg A] \rightarrow A$

A21. $\neg(A \rightarrow B) \rightarrow (A \vee \neg B)$

A22. $[\neg(A \rightarrow B) \wedge B] \rightarrow \neg B$

A23. $B \rightarrow \{[B \wedge \neg(A \rightarrow B)] \rightarrow A\}$

A24. $(A \rightarrow B) \vee \neg(A \rightarrow B)$

A25. $(\neg A \vee B) \vee \neg(A \rightarrow B)$

A26. $[(A \rightarrow B) \wedge (A \wedge \neg B)] \rightarrow \neg(A \rightarrow B)$

A27. $\neg(A \rightarrow B) \vee [(A \wedge \neg B) \rightarrow \neg(A \rightarrow B)]$

A28. $\{[\neg(A \rightarrow B) \wedge \neg A] \rightarrow \neg B\} \vee \neg B$

A29. $\{[\neg(A \rightarrow B) \wedge B] \rightarrow A\} \vee A$

3 2 set-up Routley-Meyer semantics for the Lti-logics

In the present section, a 2 set-up Routley-Meyer semantics for the Lti-logics displayed in the previous section is developed. I begin by presenting the Lti-models of interest in this paper and remarking some facts concerning the ternary relation.

Definition 3.1 (2 set-up Lti-models) A 2 set-up Lti-model (Lti-model, for short) is a structure $(K, R, *, \vDash)$ where

- K is a set which contains two elements –labelled O and O^*– and no other elements. O is the only designated set-up and O^* is its $*$-image[6].
- $*$ is an involutive unary operator defined on K such that for any $x \in K$, $x = x^{**}$.
- R is a ternary relation on K defined as follows for each i $(1 \leq i \leq 8)$: if a, b, $c \in K$, then $Rabc$ iff

  Lt1-models: $(a = O \ \& \ b = c)$ or $(a \neq b \ \& \ c = O^*)$[7].

  Lt2-models: $b = c$ or $(a = c = O^* \ \& \ b = O)$.

  Lt3-models: $(a = O \ \& \ b = c)$ or $(a = O^* \ \& \ b = O)$;

  Lt4-models: $a = b = c$ or $(c = O^* \ \& \ a \neq b)$.

  Lt5-models: $a = O^*$ or $b = c$.

  Lt6-models: $(a = O \ \& \ b = c)$ or $a = b = c$ or $(a = O^* \ \& \ b \neq c)$.

  Lt7-models: $(a = O \ \& \ b = c)$ or $(b = c = O)$ or $(a = O^* \ \& \ b \neq c)$.

  Lt8-models: $(a = O \ \& \ b = c)$ or $(a \neq O \ \& \ b \neq c)$.

- $\vDash$ is a (valuation) relation from K to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable p, wffs A, B and $a \in K$:

  (i) $a \vDash p$ or $a \nvDash p$

  (ii) $a \vDash A \wedge B$ iff $a \vDash A$ & $a \vDash B$

  (iii) $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$

  (iv) $a \vDash A \to B$ iff for all b, $c \in K$, $(Rabc \ \& \ b \vDash A) \Rightarrow c \vDash B$

  (v) $a \vDash \neg A$ iff $a^* \nvDash A$

Remark 3.2 (Ternary relations in K) Suppose $O \neq O^*$. Now, given the definition of R (cf. Definition 3.1), the following ternary relations are the only ones holding for each Lti-model $(1 \leq i \leq 8)$:

Lt1-model: $R = \{ROOO,\ ROO^*O^*,\ RO^*OO^*\}$.

Lt2-model: $R = \{ROOO,\ ROO^*O^*,\ RO^*OO^*,\ RO^*O^*O^*,\ RO^*OO\}$.

Lt3-model: $R = \{ROOO,\ ROO^*O^*,\ RO^*OO^*,\ RO^*OO\}$.

Lt4-model: $R = \{ROOO,\ ROO^*O^*,\ RO^*OO^*,\ RO^*O^*O^*\}$.

Lt5-model: $R = \{ROOO,\ ROO^*O^*,\ RO^*OO^*,\ RO^*O^*O^*,\ RO^*O^*O,\ RO^*OO\}$.

Lt6-model: $R = \{ROOO,\ ROO^*O^*,\ RO^*OO^*,\ RO^*O^*O^*,\ RO^*O^*O\}$.

Lt7-model: $R = \{ROOO,\ ROO^*O^*,\ RO^*OO^*,\ RO^*O^*O,\ RO^*OO\}$.

Lt8-model: $R = \{ROOO,\ ROO^*O^*,\ RO^*OO^*,\ RO^*O^*O\}$.

Next, we stablish the notions of truth, validity and semantic consequence.

Definition 3.3 (Truth in a class of L-models) Let L be an Lti-logic, $\mathfrak{M}$ be a class of L-models and $\mathsf{M} \in \mathfrak{M}$. A wff A is true in M iff $O \vDash A$ in this model.

[6] In contrast to general Routley-Meyer relational semantics, the 2 set-up semantics is a special kind of reduced Routley-Meyer semantics –this is, a ternary relational semantics with reduced models. Reduced models in Routley-Meyer semantics are those where the set of designated points is reduced to one element. As opposed to other reduced models, 2 set-up models are also characterized by the fact that the set of general worlds or set-ups is reduced to two elements: the designated world and its $*$-image. Concerning both types of ternary relational semantics and the relation between them, cf. [4].

[7] This clause is equivalent to Brady's clause for BN4-models (i.e., our Lt1-models): $(a \neq O$ or $b = c)$ & $[a \neq O^*$ or $(b = O \ \& \ c = O^*)]$. Cf. [6].

Definition 3.4 (Validity in a class of L-models) Let L be an Lti-logic, $\mathfrak{M}$ be a class of L-models and $\mathsf{M} \in \mathfrak{M}$. A wff A is valid in $\mathfrak{M}$ (in symbols, $\vDash A$) iff $O \vDash A$ in all $\mathsf{M} \in \mathfrak{M}$.

Definition 3.5 (Semantic consequence in a class of L-models) Let L be an Lti-logic and $\mathfrak{M}$ be a class of L-models. Then, for all $\mathsf{M} \in \mathfrak{M}$, any set of wffs Γ and wff A: $\Gamma \vDash_{\mathsf{M}} A$ (A is a semantic consequence of Γ in the model M) iff $O \vDash A$ whenever $O \vDash \Gamma$ (in particular, $O \vDash \Gamma$ iff $O \vDash B$ for all $B \in \Gamma$). Then, $\Gamma \vDash_{\mathfrak{M}} A$ (A is a semantic $\mathfrak{M}$-consequence of Γ) iff $\Gamma \vDash_{\mathsf{M}} A$ for all $\mathsf{M} \in \mathfrak{M}$.

Proposition 3.6 ($O^* \vDash \neg A$ iff $O \nvDash A$) Let L be an Lti-logic. For any L-model M and wff A, $O^* \vDash \neg A$ iff $O \nvDash A$.

Proof. By clause (v) in Definition 3.1 and the involutiveness of $*$. ■

Lemma 3.7 (Entailment lemma) Let L be an Lti-logic ($1 \leq i \leq 8$), for any wffs A, B, $\vDash A \to B$ iff ($a \vDash A \Rightarrow a \vDash B$ for all $a \in K$) in all L-models.

Proof. This proof is completely similar for any of the Lti-logics ($1 \leq i \leq 8$) and can be found in the literature (cf. [15, Lemma 3.8]).

Lemma 3.8 (Soundness of the Lti-logics) Let L be an Lti-logic ($1 \leq i \leq 8$), for any set of wffs Γ and wff A, if $\Gamma \vdash_{Lti} A$, then $\Gamma \vDash_{Lti} A$.

Proof. We have to prove three cases: (I) $A \in \Gamma$; (II) the rules preserve L-validity; (III) The axioms A1-A29 are valid in the corresponding Lti-logic. By using the labels (i)-(v) throughout the following proofs, we shall refer to the clauses in Definition 3.1. Firstly, the proof of case (I) is trivial. Let us prove now case (II). The proofs of cases when A is derived by ADJ and MP are displayed in [15]. It remains to prove the cases when A is derived by dMP, dSUF, dPREF, dCON and dCTE. A couple of instances will suffice as an illustration. Let us prove the cases when A is derived by (a) dSUF and (b) dCTE.

(a) A is derived by dCTE. The proof is identical for any of the Lti-logics considered in this paper. Suppose $\Gamma \vDash D \vee (B \wedge \neg C)$ and $O \vDash \Gamma$. Then, we have (1) $O \vDash D \vee (B \wedge \neg C)$ by Definition 3.5. Further, as reductio hypothesis, suppose (2) $O \nvDash D \vee \neg(B \to C)$, therefore (3) $O \nvDash D$ and (4) $O \nvDash \neg(B \to C)$ (i.e., $O^* \vDash B \to C$) by (iii). On the other hand, given (1), we have (5) $O \vDash D$ or (6) $O \vDash B \wedge \neg C$. Therefore (7) $O \vDash B \wedge \neg C$ (i.e., $O \vDash B$ and $O^* \nvDash C$), given (3). Lastly, by (4), (7) and the postulate RO^*OO^* –which is shared by all the Lti-logics (cf. Remark 3.2)–, we get (8) $O^* \vDash C$, contradicting (7).

(b) A is derived by dSUF. We shall display the proof for the logic Lt5 (i.e., E4) and then provide a few comments on how it could be easily adapted to suit the rest of the Lti-logics. As in the previous case, we assume $\Gamma \vDash E \vee (B \to C)$ and $O \vDash \Gamma$, therefore, by Definition 3.5, (1) $O \vDash E \vee (B \to C)$ (i.e., $O \vDash E$ or $O \vDash B \to C$). We also suppose (2) $O \nvDash E \vee [(C \to D) \to (B \to D)]$ by reductio, this is, (3) $O \nvDash E$ and (4) $O \nvDash (C \to D) \to (B \to D)$. Given (1) and (3), we clearly have (5) $O \vDash B \to C$. Then, by applying clause (iv) to (4), there are x, $y \in K$ such that $ROxy$, $x \vDash C \to D$ and $y \nvDash B \to D$. Now, given the definition of R in Lt5 (cf. Remark 3.2), we have to consider the two following alternatives (6) and (7): (6i) $ROOO$, (6ii) $O \vDash C \to D$, (6iii) $O \nvDash B \to D$ or (7i) ROO^*O^*, (7ii) $O^* \vDash C \to D$, (7iii) $O^* \nvDash B \to D$. Suppose (6), then there are z, $w \in K$ such that $ROzw$, $z \vDash B$ and $w \nvDash D$. Now, we have two alternatives again: (8i) $ROOO$, (8ii) $O \vDash B$, (8iii) $O \nvDash D$ or (9i) ROO^*O^*, (9ii) $O^* \vDash B$ and (9iii) $O^* \nvDash D$. Let us first consider (8), by applying clause (iv) twice, we first get (10) $O \vDash C$ –given (5), (8i) and (8ii)– and therefore, (11) $O \vDash D$ –given (6ii) and (8i)–, which contradicts (8iii). Similarly, when we suppose (9), we get first (12) $O^* \vDash C$ –given (5), (9i) and (9ii)– and in consequence, (13) $O^* \vDash D$ –given (6ii) and (9i)–, contradicting (9iii). Next, we have to consider case (7) above. Given (7iii), there are z, $w \in K$ such that RO^*zw, $z \vDash B$ and $w \nvDash D$. Thus, we have to consider four different alternatives in this case (cf. the definition of R for the system Lt5 displayed in Remark 3.2): (14i) RO^*OO^*, (14ii) $O \vDash B$, (14iii) $O^* \nvDash D$; (15i) RO^*OO, (15ii) $O \vDash B$, (15iii) $O \nvDash D$; (16i) $RO^*O^*O^*$, (16ii) $O^* \vDash B$, (16iii) $O^* \nvDash D$; (17i) RO^*O^*O, (17ii) $O^* \vDash B$, (17iii) $O \nvDash D$. Now, a contradiction can easily be found in each of these cases following the preceding method. However, it is worth noting that for some of these cases postulates $ROOO$ or ROO^*O^* are also needed –which entails no additional difficulty since they both are included in every Lti-model (cf. Remark 3.2). As for the rest of the Lti-logics, omitting some steps from the given proof will be sufficient. For instance, in order to prove the fact that rule dSUF preserves validity in the logic Lt1 (i.e., BN4), there is no need to consider cases (15), (16) and (17) above since the definition of R in BN4 does not include the postulates of these cases. Something similar can be said for the rest of the Lti-logics[8].

(III) The axioms A1-A12 are valid in all the Lti-logics ($1 \leq i \leq 8$) and the axioms A13-A29 are valid in the corresponding Lti-logics (cf. Definition 2.7). We shall prove a few instances as an illustration. On the one hand, A1-A8 are classic relevant axioms and their proofs are similar to those given for general RM-semantics (cf. [22, chapter 4]). The validity of A9 is proven as in [15] and A10 can be proven in a similar way. Let us now prove a few other axioms as examples. The use of the Entailment Lemma (cf. Lemma 3.7) will simplify the proofs of axioms whose main connective is a conditional.

A12 $(A \to B) \vee [(\neg A \wedge B) \to (A \to B)]$ is valid in every Lti-logic ($1 \leq i \leq 8$). For some model in any Lti-logic, suppose (1) $O \nvDash (A \to B) \vee [(\neg A \wedge B) \to (A \to B)]$ as the reductio hypothesis. Then, (2) $O \nvDash (A \to B)$ and (3) $O \nvDash (\neg A \wedge B) \to (A \to B)$. By (3), there are $x, y \in K$ such that $ROxy$, $x \vDash \neg A \wedge B$, $y \nvDash A \to B$. We have to consider two cases: (4i) $ROOO$, (4ii) $O \vDash \neg A \wedge B$ (i.e., $O^* \nvDash A$ and $O \vDash B$), (4iii) $O \nvDash A \to B$ and (5i) ROO^*O^*, (5ii) $O^* \vDash \neg A \wedge B$ (i.e., $O \nvDash A$ and $O^* \vDash B$), (5iii) $O^* \nvDash A \to B$. Let us first consider (4). Proceeding as before, we have that there are $z, w \in K$ such that $ROzw$, $z \vDash A$, $w \nvDash B$. Therefore, we face two more alternatives: (6i) $ROOO$, (6ii) $O \vDash A$, (6iii) $O \nvDash B$ and (7i) ROO^*O^*, (7ii) $O^* \vDash A$, (7iii) $O^* \nvDash B$. However, (4ii) contradicts (6iii) and (7ii). Let us now suppose (5). On the other hand, we also have for some $a, b \in K$, $ROab$, $a \vDash A$, $b \nvDash B$, given (2). Again, there are only a couple of possibilities: (8i) $ROOO$, (8ii) $O \vDash A$, (8iii) $O \nvDash B$ or (9i) ROO^*O^*, (9ii) $O^* \vDash A$, (9iii) $O^* \nvDash B$. However, (5ii) contradicts (8ii) and (9iii).

A18 $A \to [B \vee \neg(A \to B)]$ is valid in the Lti-logics such that $i \in \{2, 3, 5, 7\}$. Two cases must be considered. Let L be one of the aforementioned Lti-logics, for an arbitrary L-model, (a) $O \vDash A \Rightarrow O \vDash B \vee \neg(A \to B)$ and (b) $O^* \vDash A \Rightarrow O^* \vDash B \vee \neg(A \to B)$. Case (a): By reductio, suppose (1) $O \vDash A$ but (2) $O \nvDash B \vee \neg(A \to B)$ (i.e., $O \nvDash B$ and $O^* \vDash A \to B$). Given (1), (2) and the definition of R for the considered Lti-logics (cf. Definition 3.1 and Remark 3.2), we use (3) RO^*OO and clause (iv) in Definition 3.1 to obtain (4) $O \vDash B$, which contradicts (2). Case (b): we proceed similarly but use postulate ROO^*O^* (valid in Lt2, Lt3, Lt5 and Lt7) instead.

A22 $[\neg(A \to B) \wedge B] \to \neg B$ is valid in the Lti-logics such that $i \in \{2, 3, 5, 6, 7, 8\}$. As in A18, we have to consider two cases[9]. Case (a): suppose (1) $O \vDash \neg(A \to B) \wedge B$ (i.e., $O^* \nvDash A \to B$ and $O \vDash B$) but (2) $O \nvDash \neg B$ (i.e., $O^* \vDash B$). Then, there are $x, y \in K$ such that RO^*xy, $x \vDash A$ and $y \nvDash B$. Now, the number of cases we face depends on how R is defined in each particular Lti-model. Let us consider by way of example the case for Lt5-models. In this case, we face four different alternatives: (3i) $RO^*O^*O^*$, (3ii)$O^* \vDash A$, (3iii) $O^* \nvDash B$; (4i) RO^*O^*O, (4ii)$O^* \vDash A$, (4iii) $O \nvDash B$; (5i) RO^*OO^*, (5ii)$O \vDash A$, (5iii) $O^* \nvDash B$; (6i) RO^*OO, (6ii) $O \vDash A$, (6iii) $O \nvDash B$. However, every alternative (iii) contradicts either (1) or (2). Case (b): suppose (1) $O^* \vDash \neg(A \to B) \wedge B$ (i.e., $O \nvDash A \to B$ and $O^* \vDash B$) but (2) $O^* \nvDash \neg B$ (i.e., $O \vDash B$). Then, there are $x, y \in K$ such that $ROxy$, $x \vDash A$ and $y \nvDash B$. Then, we only have to consider two alternatives: (3) $ROOO$, $O \vDash A$, $O \nvDash B$; (4) ROO^*O^*, $O^* \vDash A$, $O^* \nvDash B$. Again, these alternatives contradict (2) and (1), respectively.

A25 $(\neg A \vee B) \vee \neg(A \to B)$ is valid in the Lti-logics such that $i \in \{5, 6, 7, 8\}$. For some model in any of the Lti-logics belonging to the previous subset, suppose (1) $O \nvDash (\neg A \vee B) \vee \neg(A \to B)$ by reductio, this is, (2) $O \nvDash \neg A \vee B$ (i.e., $O^* \vDash A$ and $O \nvDash B$) and (3) $O \nvDash \neg(A \to B)$ (i.e., $O^* \vDash A \to B$). Then, by the definition of R in the models of the considered systems (cf. Remark 3.2), we have (4) RO^*O^*O. Finally, we get $O \vDash B$ –thus, contradicting (3)– by clause (iv) given (2), (3) and (4). ■

4 Useful preliminary Lemmas to the Completeness Theorem

First of all, we set out some preliminary definitions concerning the notions of theories and the classes of theories considered in this paper. We underline that the label EL will be used throughout this section to refer to an extension of the logic L (cf. Definition 2.6). In particular, the following notions are displayed in general for any Eb4-logic –i.e., for any extension of the logic b4– with no additional (primitive) rules to those of b4 (cf. Definition 2.3). It is worth noting that we shall only provide a

[8] In order to check which steps should be omitted for the corresponding proofs in the rest of the Lti-logics, the reader is advised to review Remark 3.2. We shall find alike proof outlines throughout this paper and proceed in a similar manner.

[9] Similar to the proof of dSUF, case (i) is different for each of the considered Lti-logics depending on how R is defined in each of the Lti-models (cf. Remark 3.2). We proceed as done in the proof of suffixing, this is, we display the proof of case (i) for the logic Lt5. Adapting this proof to any other of the Lti-logics ($i \in \{2, 3, 5, 6, 7, 8\}$) is trivial.

brief outline of the proofs for the propositions displayed in this section since the complete proofs for the very same range of logics are already available (cf. [10]).

Definition 4.1 (Eb4-theories) Let L be an Eb4-logic. An L-theory a is a set of wffs closed under Adjunction (Adj) and provable L-entailment (L-ent). That is to say, a set of wffs is closed under Adj iff, whenever A, $B \in a$, then $A \wedge B \in a$; a set of wffs is closed under L-ent iff, whenever $A \rightarrow B$ is a theorem of L and $A \in a$, then $B \in a$.

Definition 4.2 (Types of Eb4-theories) Let L be an Eb4-logic and a an L-theory. We set (1) a is prime iff, for wffs A and B, whenever $A \vee B \in a$, then either $A \in a$ or $B \in a$; (2) a is regular iff a contains all theorems in L; (3) a is trivial iff it contains every wff; (4) a is empty iff it contains no wff.

Definition 4.3 (Sets of wffs closed under a certain rule) A set of wffs Γ is closed under a rule r iff the conclusion of r belongs to Γ whenever the hypothesis of r belongs to Γ.

Definition 4.4 (Full regularity) Let L be an Eb4-logic, an L-theory a is fully regular iff it is a regular L-theory (cf. Definitions 4.1 and 4.2) which is closed under the rules of b4 (i.e., MP, dMP, dCON, dPREF, dSUF, dCTE; cf. Definition 4.3).

Next, we note that fully regular L-theories are closed under the derived rules of b4 (i.e., "the non-disjunctive rules").

Proposition 4.5 (Derived rules under which fully regular Eb4-theories are closed) Let L be an Eb4-logic, if a is a fully regular L-theory, then it is closed under (1) CON, (2) PREF, (3) SUF, (4) CTE, (5) MT and (6) TRAN.

PROOF. Cases (1)-(4): by A4 and T1 ($A \leftrightarrow (A \vee A)$) and the fact that a is fully regular (i.e., closed under dCON, dPREF, dSUF and dCTE, respectively for each case). Cases (5)-(6): by hypothesis, a is fully regular (therefore closed under MP) and by the fact that a is closed under CON and SUF (given what has already been proved in cases (1) and (3)), respectively for each case. ■

In what follows, we shall introduce the extension lemmas. These lemmas are essential to the completeness theorem and are developed here following Routley *et al.*'s method [22, Chapter 4] as applied in [6, 14, 17]. Firstly, we note the following definition.

Definition 4.6 (Disjunctive Eb4-derivability) Let L be an Eb4-logic, Γ and Θ be non-empty sets of wffs, Θ is disjunctively derivable from Γ in Eb4 (in symbols, $\Gamma \vdash^d_L \Theta$) iff $A_1 \wedge ... \wedge A_n \vdash_L B_1 \vee ... \vee B_n$ for some wffs $A_1, ..., A_n \in \Gamma$ and $B_1, ..., B_n \in \Theta$.

The following is a necessary lemma to prove the *Extension to maximal sets* lemma (Lemma 4.9).

Lemma 4.7 (Preliminary lemma to the extension lemma) Let L be an Eb4-logic closed under no other rules than those specified in Definition 2.3. For any wffs A, B_1,..., B_n, if $\{B_1, ..., B_n\} \vdash_L A$, then, for any wff C, $C \vee (B_1 \wedge ... \wedge B_n) \vdash_L C \vee A$.

PROOF. Induction on the length of A (cf. p. 27 in [6] and also Lemma 6.2 in [17] or Lemma 7.3 in [14]). ■

Now, the process of extending sets of wffs to maximal sets is required.

Definition 4.8 (Maximal sets) Let L be an Eb4-logic, Γ is an L-maximal set of wffs iff $\Gamma \nvdash^d_L \overline{\Gamma}$ ($\overline{\Gamma}$ is the complement of Γ).

Lemma 4.9 (Extension to maximal sets) Let L be an Eb4-logic closed under no other rules than those specified in Definition 2.3. Let Γ and Θ be sets of wffs such that $\Gamma \nvdash^d_L \Theta$. Then, there are sets of wffs Γ' and Θ' such that $\Gamma \subseteq \Gamma'$, $\Theta \subseteq \Theta'$, $\Theta' = \overline{\Gamma'}$ and $\Gamma' \nvdash^d_L \Theta'$ (i.e., Γ' is an L-maximal set such that $\Gamma' \nvdash^d_L \Theta'$).

Proof. Cf. Lemma 9 in [6] and also Lemma 6.4 in [17] or Lemma 7.4 in [14]. ■

Finally, the *Primeness* Lemma is proved.

Lemma 4.10 (Primeness) Let L be an Eb4-logic closed under no other rules than those specified in Definition 2.3. If Γ is an L-maximal set, then it is a fully regular prime L-theory.

Proof. This Lemma was already provided for the exact same frame of logics in [10, Lemma 7.5]. ■

In order to prove the completeness theorem in Section 5, the following items are also required. First of all, we recall the notion of "the set of consequences of a given set of wffs Γ in a logic L".

Definition 4.11 (The set $Cn\Gamma[L]$) Let L be an L*ti*-logic, the set of consequences in L of a set of wffs Γ (in symbols $Cn\Gamma[L]$) is defined as follows: $Cn\Gamma[L] = \{A \mid \Gamma \vdash_L A\}$.

In relation to the previous definition, we note the following remark.

Remark 4.12 (The set of consequences of Γ in any Lt*i*-logic is a fully regular theory) Let L be an L*ti*-logic. Then, it is obvious that for any Γ, $Cn\Gamma[L]$ contains all theorems of L and is closed under the rules of L. Consequently, it is also closed under L-entailment.

To conclude this section, we build the regular prime L-theory $\mathcal{T}$ upon which the canonical model is defined.

Proposition 4.13 (The building of $\mathcal{T}$) Let L be an L*ti*-logic, Γ a set of wffs and A a wff such that $\Gamma \nvdash_L A$. Then, there is a fully regular, prime L-theory $\mathcal{T}$ such that $\Gamma \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$.

Proof. Suppose $\Gamma \nvdash_L A$ (i.e., $A \notin Cn\Gamma[L]$ given Remark 4.1). Then, $Cn\Gamma[L] \nvdash^d_L \{A\}$ by Definition 4.6; otherwise $(B_1 \wedge ... \wedge B_n) \vdash_L A$ for some B_1, ..., $B_n \in \Gamma$ and hence A would be in $Cn\Gamma[L]$ after all. Next, there is some (fully regular) prime L-theory $\mathcal{T}$ such that $\Gamma \subseteq \mathcal{T}$ (since $\Gamma \subseteq Cn\Gamma[L]$) and $A \notin \mathcal{T}$, by application of Lemmas 4.9 and 4.10. ■

5 Completeness of the Lt*i*-logics

In this section, the completeness theorem for the L*ti*-logics is provided. We first note some preliminary notions and then the canonical model is defined upon the theory $\mathcal{T}$ built at the end of Section 4.

Definition 5.1 (R^P, $*^P$and $\vDash^P$) Let K^P be the set of all prime theories. Then, R^P, $*^P$ and $\vDash^P$ are defined as follows for all a, b, $c \in K^P$ and wffs A, B: (i) $R^P abc$ iff $(A \to B \in a \ \& \ B \in b) \Rightarrow B \in c$; (ii) $a^{*^P} = \{A \mid \neg A \notin a\}$; (iii) $a \vDash^P A$ iff $A \in a$.

Next, we prove that $*^P$ is an operation on K^P.

Proposition 5.2 ($*^P$ is an operation on K^P) (1) Let $a \in K^P$. Then, $a^{*^P} \in K^P$ as well. (2) For any wff A, $\neg A \in a^{*^P}$ iff $A \notin a$.

Proof. Cf. [22, Chapter 4]. (1) We get that a^* is closed under L-ent by the rule CON and the fact that a is closed under L-ent; a^* is closed under Adj by T2; a^* is prime by T3. (2) By A7 and A8. ■

In what follows, the canonical model is defined upon the fully regular and prime L-theory $\mathcal{T}$ built in Proposition 4.13.

Definition 5.3 (The canonical L-model) Let L be an Lti-logic, the canonical L-model is the structure $< K^C, R^C, *^C, \vDash^C >$, where $K^C = \{\mathcal{T}, \mathcal{T}^{*^C}\}$ and $\mathcal{T}$ is the fully regular and prime L-theory built in Proposition 4.13. Additionally, $*^C$, R^C and $\vDash^C$ are the restrictions of $*^P$, R^P and $\vDash^P$ to the set K^C.

Let L be an Lti-logic, the canonical L-model will be shown to be an L-model by means of which non-theorems of L are falsified. From now on, the subscripts P and C are omitted above $*$ and R when there is no risk of confusion. First, we show that $*$ is an involution on K.

Proposition 5.4 ($a = a^{**}$) For any $a \in K^P$, $a = a^{*^P *^P}$.
PROOF. Immediate by A7, A8 and closure of a under L-ent. ■

Corollary 5.5 ($*^C$ is an involutive operation on K^C) The operation $*^C$ is an involutive operation on K^C, this is, for any $a \in K^C$, $a^{*^C} \in K^C$ and moreover, $a = a^{*^C *^C}$.
PROOF. Immediate by Proposition 5.2 and Proposition 5.4. ■

Finally, it remains to prove that the ternary relation R and the valuation clauses in Definition 3.1 hold canonically.

Lemma 5.6 (R holds canonically in each Lti-model) If $a, b, c \in K^C$, then $R^C abc$ iff
Lt1-models: $(a = \mathcal{T}\ \&\ b = c)$ or $(a \neq b\ \&\ c = \mathcal{T}^{*^C})$.
Lt2-models: $b = c$ or $(a = c = \mathcal{T}^{*^C}\ \&\ b = \mathcal{T})$.
Lt3-models: $(a = \mathcal{T}\ \&\ b = c)$ or $(a = \mathcal{T}^{*^C}\ \&\ b = \mathcal{T})$.
Lt4-models: $a = b = c$ or $(c = \mathcal{T}^{*^C}\ \&\ a \neq b)$.
Lt5-models: $a = \mathcal{T}^{*^C}$ or $b = c$.
Lt6-models: $(a = \mathcal{T}\ \&\ b = c)$ or $a = b = c$ or $(a = \mathcal{T}^{*^C}\ \&\ b \neq c)$.
Lt7-models: $(a = \mathcal{T}\ \&\ b = c)$ or $(b = c = \mathcal{T})$ or $(a = \mathcal{T}^{*^C}\ \&\ b \neq c)$.
Lt8-models: $(a = \mathcal{T}\ \&\ b = c)$ or $(a \neq \mathcal{T}\ \&\ b \neq c)$.

PROOF. Given Remark 3.2 and Corollary 5.5, a subset of the following relations have to be proven for each Lti-logic ($1 \leq i \leq 8$): (a) $R\mathcal{T}\mathcal{T}\mathcal{T}$; (b) $R\mathcal{T}\mathcal{T}^*\mathcal{T}^*$; (c) $R\mathcal{T}^*\mathcal{T}\mathcal{T}^*$; (d) $R\mathcal{T}^*\mathcal{T}^*\mathcal{T}^*$; (e) $R\mathcal{T}^*\mathcal{T}\mathcal{T}$; (f) $R\mathcal{T}^*\mathcal{T}^*\mathcal{T}$. It is worth underlining that $\mathcal{T}$ is closed under the rules of b4 since it is fully regular (cf. Definition 5.3).

(a) $R\mathcal{T}\mathcal{T}\mathcal{T}$ holds in any canonical Lti-model ($1 \leq i \leq 8$). For wffs A, B, suppose $A \to B \in \mathcal{T}$ and $A \in \mathcal{T}$. Then, $B \in \mathcal{T}$ follows immediately given that $\mathcal{T}$ is closed under MP, as it was required.

(b) $R\mathcal{T}\mathcal{T}^*\mathcal{T}^*$ holds in any canonical Lti-model ($1 \leq i \leq 8$). For wffs A, B, suppose (1) $A \to B \in \mathcal{T}$ and (2) $A \in \mathcal{T}^*$ (i.e., $\neg A \notin \mathcal{T}$). Now, we get (3) $\neg B \to \neg A \in \mathcal{T}$ by the fact that $\mathcal{T}$ is closed under CON. Then, (4) $\neg B \notin \mathcal{T}$ (i.e, $B \in \mathcal{T}^*$) by 2, 3 and the closure of $\mathcal{T}$ under MP.

(c) $R\mathcal{T}^*\mathcal{T}\mathcal{T}^*$ holds in any canonical Lti-model ($1 \leq i \leq 8$). For wffs A, B, suppose (1) $A \to B \in \mathcal{T}^*$ (i.e., $\neg(A \to B) \notin \mathcal{T}$) and (2) $A \in \mathcal{T}$. Furthermore, suppose (3) $B \notin \mathcal{T}^*$ (i.e., $\neg B \in \mathcal{T}$) by reductio. Then, (4) $A \wedge \neg B \in \mathcal{T}$ given that $\mathcal{T}$ is closed by Adj and therefore, we also get (5) $\neg(A \to B) \in \mathcal{T}$ by the closure of $\mathcal{T}$ under CTE. However, 5 contradicts 1.

(d) $R\mathcal{T}^*\mathcal{T}^*\mathcal{T}^*$ holds in the canonical Lti-model such that $i \in \{2, 4, 5, 6\}$. For wffs A, B, suppose (1) $A \to B \in \mathcal{T}^*$ (i.e., $\neg(A \to B) \notin \mathcal{T}$) and (2) $A \in \mathcal{T}^*$ (i.e., $\neg A \notin \mathcal{T}$). Then, we have (3) $\neg A \vee \neg(A \to B) \notin \mathcal{T}$ by the primeness of $\mathcal{T}$. Finally, by A19 ($\neg B \to [\neg A \vee \neg(A \to B)]$) and that $\mathcal{T}$ is closed under L-ent, (4) $\neg B \notin \mathcal{T}$ (i.e., $B \in \mathcal{T}^*$).

(e) $R\mathcal{T}^*\mathcal{T}\mathcal{T}$ holds in the canonical Lti-model such that $i \in \{2, 3, 5, 7\}$. For wffs A, B, suppose (1) $A \to B \in \mathcal{T}^*$ (i.e., $\neg(A \to B) \notin \mathcal{T}$) and (2) $A \in \mathcal{T}$. By A18 ($A \to [B \vee \neg(A \to B)]$) and 2, we get $B \vee \neg(A \to B) \in \mathcal{T}$, this is, $B \in \mathcal{T}$ or $\neg(A \to B) \in \mathcal{T}$, by primeness of $\mathcal{T}$, whence $B \in \mathcal{T}$ given 1, as it was to be proved.

(f) $R\mathcal{T}^*\mathcal{T}^*\mathcal{T}$ holds in the canonical Lti-model such that $i \in \{5, 6, 7, 8\}$. For wffs A, B, suppose (1) $A \to B \in \mathcal{T}^*$ (i.e., $\neg(A \to B) \notin \mathcal{T}$) and (2) $A \in \mathcal{T}^*$ (i.e., $\neg A \notin \mathcal{T}$). By A25 ($(\neg A \vee B) \vee \neg(A \to B)$), we get (3) $\neg A \vee B \in \mathcal{T}$ or $\neg(A \to B) \in \mathcal{T}$ since $\mathcal{T}$ is regular and prime. Thus, (4) $\neg A \vee B \in \mathcal{T}$ (given 1), i.e., $\neg A \in \mathcal{T}$ or $B \in \mathcal{T}$. Whence, by 2, we have $B \in \mathcal{T}$. ■

Lemma 5.7 (Clauses (i)-(v) hold canonically) Let L be an Lti-logic ($1 \leq i \leq 8$). Clauses (i)-(v) in Definition 3.1 are satisfied by the canonical L-model.

Proof. The reader is advised to cf. Definitions 5.1 and 5.3. Clause (i) is immediate; clauses (ii), (iii), (v) and (iv) from left to right are proved similarly as in [15, Lemma 4.13]. Let us now prove clause (vi) from right to left. Firsly, we have to consider two situations: the first component in the ternary relation is (I) $\mathcal{T}$ or it is (II) $\mathcal{T}^*$.

(I) For some wffs A and B, suppose $A \to B \notin \mathcal{T}$ as hypothesis. Given $R\mathcal{T}\mathcal{T}\mathcal{T}$ and $R\mathcal{T}\mathcal{T}^*\mathcal{T}^*$ (cf. Remark 3.2), we have to prove: ($A \in \mathcal{T}$ & $B \notin \mathcal{T}$) or ($A \in \mathcal{T}^*$ & $B \notin \mathcal{T}^*$). By reductio, suppose that this is not the case. Then, we obtain the following alternatives: (a) $A \notin \mathcal{T}$ & $A \notin \mathcal{T}^*$; (b) $A \notin \mathcal{T}$ & $B \in \mathcal{T}^*$; (c) $B \in \mathcal{T}$ & $A \notin \mathcal{T}^*$; (d) $B \in \mathcal{T}$ & $B \in \mathcal{T}^*$. Next, we prove that none of these alternatives is possible. In particular, we will get $A \to B \in \mathcal{T}$ from each of them. Thus, a contradiction follows from each alternative and therefore case (I) will be proved. For each alternative, we will use some of the properties of $\mathcal{T}$ (cf. Proposition 4.13) and one of the axioms of b4:

(a) $A \notin \mathcal{T}$ & $A \notin \mathcal{T}^*$ (i.e., $\neg A \in \mathcal{T}$) by A9 ($\neg A \to [A \vee (A \to B)]$) and the fact that $\mathcal{T}$ is closed under L-ent and by its primeness.

(b) $A \notin \mathcal{T}$ & $B \in \mathcal{T}^*$ (i.e., $\neg B \notin \mathcal{T}$) by A11 ($(A \vee \neg B) \vee (A \to B)$) and the regularity and primeness of $\mathcal{T}$.

(c) $B \in \mathcal{T}$ & $A \notin \mathcal{T}^*$ (i.e., $\neg A \in \mathcal{T}$) By A12 ($(A \to B) \vee [(\neg A \wedge B) \to (A \to B)]$), the main hypothesis ($A \to B \notin \mathcal{T}$) and the full regularity and primeness of $\mathcal{T}$.

(d) $B \in \mathcal{T}$ & $B \in \mathcal{T}^*$ (i.e., $\neg B \notin \mathcal{T}$) by A10 ($B \to [\neg B \vee (A \to B)]$) and the fact that $\mathcal{T}$ is a prime L-theory.

(II) For some wffs A and B, suppose $A \to B \notin \mathcal{T}^*$ (i.e., $\neg(A \to B) \in \mathcal{T}$) as hypothesis. The proof is similar to that of case (I). However, we have to prove different alternatives in each Lti-logic ($1 \leq i \leq 8$) given the Definition of R (cf. Remark 3.2). A sketch of the proof is provided for each one of the considered logics. In these proofs, some of the properties of $\mathcal{T}$ –named, that it is a fully regular and prime L-theory (cf. Definitions 4.1-4.4)– along with some axioms of the Lti-logics are needed (cf. Definition 2.7).

Lt1: We have to prove $A \in \mathcal{T}$ and $B \notin \mathcal{T}^*$. By reductio, we get the following alternatives: (a) $A \notin \mathcal{T}$ or (b) $B \in \mathcal{T}^*$. Now, a contradiction is reached for each case using the main hypothesis ($A \to B \notin \mathcal{T}^*$) and axioms A14 ($A \vee [\neg(A \to B) \to A]$) and A15 ($\neg B \vee [\neg(A \to B) \to \neg B]$), respectively.

Lt2: We must prove ($A \in \mathcal{T}$ and $B \notin \mathcal{T}^*$) or ($A \in \mathcal{T}^*$ and $B \notin \mathcal{T}^*$) or ($A \in \mathcal{T}$ and $B \notin \mathcal{T}$). By reductio, we obtain eight different alternatives reducible to the following ones: (a) $A \notin \mathcal{T}$ & $A \notin \mathcal{T}^*$ or (b) $A \notin \mathcal{T}$ & $B \in \mathcal{T}^*$ or (c) $B \in \mathcal{T}$ & $B \in \mathcal{T}^*$. From each alternative, we can get a contradiction by means of A20 ($[\neg(A \to B) \wedge \neg A] \to A$), A21 ($\neg(A \to B) \to (A \vee \neg B)$) and A22 ($[\neg(A \to B) \wedge B] \to \neg B$), respectively.

Lt3: We have to prove ($A \in \mathcal{T}$ and $B \notin \mathcal{T}^*$) or ($A \in \mathcal{T}$ and $B \notin \mathcal{T}$). By reductio, we have (a) $A \in \mathcal{T}$ or (b) $A \notin \mathcal{T}$ & $B \in \mathcal{T}^*$ or (c) $B \in \mathcal{T}^*$ & $B \in \mathcal{T}$ or (d) $A \notin \mathcal{T}$ & $B \in \mathcal{T}$. Then, a contradiction follows from alternatives (a), (b) and (c) given the recently mentioned axioms A14, A21 and A22 and the same can be said about alternative (d) given axiom A23 ($B \to [B \wedge \neg(A \to B)] \to A]$).

Lt4: We must prove ($A \in \mathcal{T}$ and $B \notin \mathcal{T}^*$) or ($A \in \mathcal{T}^*$ and $B \notin \mathcal{T}^*$). By reductio, we face four different alternatives: (a) $A \notin \mathcal{T}$ & $A \notin \mathcal{T}^*$ or (b) $A \notin \mathcal{T}$ & $B \in \mathcal{T}^*$ or (c) $B \in \mathcal{T}^*$ & $A \notin \mathcal{T}^*$ or (d) $B \in \mathcal{T}^*$. Case (a) can be solved by A20 and cases (b), (c) and (d) by A15. [We could also use A21 instead of A15 in case (b)].

Lt5: In this case, A20 and A22 can be used. However, the proof for Lt5 is omitted here since is already available in the literature (cf. [15]).

Lt6: We have to prove ($A \in \mathcal{T}$ and $B \notin \mathcal{T}^*$) or ($A \in \mathcal{T}^*$ and $B \notin \mathcal{T}^*$) or ($A \in \mathcal{T}^*$ and $B \notin \mathcal{T}$). By reductio, we obtain eight different alternatives reducible to three: the alternatives (a) and (c) in Lt2, which can be solved likewise; and $A \notin \mathcal{T}^*$ & $B \in \mathcal{T}^*$, for which we can use A28 ($\{[\neg(A \to B) \wedge \neg A] \to \neg B\} \vee \neg B$).

Lt7: We must prove ($A \in \mathcal{T}$ and $B \notin \mathcal{T}^*$) or ($A \in \mathcal{T}$ and $B \notin \mathcal{T}$) or ($A \in \mathcal{T}^*$ and $B \notin \mathcal{T}$). We proceed similarly as before and obtain four alternatives from which we find a contradiction using one of the following axioms: A20, A22 and A29 ($\{[\neg(A \to B) \wedge B] \to A\} \vee A$).

Lt8: What we have to prove in this case is ($A \in \mathcal{T}$ and $B \notin \mathcal{T}^*$) or ($A \in \mathcal{T}^*$ and $B \notin \mathcal{T}$). By applying the same method, we face four different alternatives that can be solved by means of A20, A22, A28 and A29 as in previous cases. ■

We can now prove completeness of the Lti-logics. First, we show that the canonical Lti-model is indeed an Lti-model.

Lemma 5.8 (The canonical Lti-model is an Lti-model) Let L be an Lti-logic, then the canonical L-model is indeed an L-model.

Proof. On the one hand, we know that $*^C$ is an involution on K^C (Corollary 5.5). On the other hand, the ternary relation R^C holds canonically in any L-model (Lemma 5.6). Finally, clauses (i)-(v) hold canonically given Lemma 5.7. ■

Theorem 5.9 (Completeness of the Lti-logics) Let L be an Lti-logic ($1 \leq i \leq 8$). For any set of wffs Γ and wff A, if $\Gamma \vDash_L A$, then $\Gamma \vdash_L A$.

Proof. We prove Theorem 5.9 by contraposition. Let L be an Lti-logic ($1 \leq i \leq 8$), then for some set of wffs Γ and wff A, suppose $\Gamma \nvdash_L A$. By proposition 4.13, there is a fully regular prime L-theory $\mathcal{T}$ such that $\Gamma \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. Then, the canonical L-model is defined upon $\mathcal{T}$ as in Definition 5.3 and by Lemma 5.8, the canonical L-model is indeed an L-model. Finally, $\Gamma \nvDash^C A$ since $\mathcal{T} \vDash^C \Gamma$ but $\mathcal{T} \nvDash^C A$. Therefore, $\Gamma \nvDash_L A$ by Definition 3.5. ■

6 Conclusion

In [10], six different logics were developed as possible companions to the system BN4 and were studied from the perspective of a bivalent Belnap-Dunn type semantics. These systems are the logics determined by the class of all implicative expansions of Belnap's matrix MB4 verifying B while maintaining the conditional structure of MBN4 or ME4.

In the present paper, we have offered a different perspective to deepen our knowledge of these logics. In particular, these new systems –the Lti-logics–had been endowed with a 2 set-up ternary relational semantics following the strategies applied to BN4 by Brady (cf. [6]). Thus, the research regarding the companions to BN4 is now completed because we have already provided the very same semantic tools (both a bivalent Belnap-Dunn type semantics and a 2 set-up ternary relational semantics [10]) to analyse these logics that Brady himself applied to study its system BN4. Finally, we may indeed claim –as he hoped when that paper was published–that his method of proof is fairly general and other model structures can be axiomatized by appropriate modifications to the proofs he provided ([6], p. 9).

Acknowledgements

This work is supported by the Spanish Ministry of Science and Innovation (MCIN/AEI/10.13039/50110 0011033) under Grant PID2020-116502GB-I00. Thanks are due to José Manuel Méndez for useful comments on a draft version of this paper. I also thank an anonymous referee of the AJL for their useful comments and suggestions on a previous version of this paper.

References

[1] Alan Ross Anderson and Nuel D. Belnap. Entailment: The Logic of Relevance and Necessity. Vol. I. Princeton University Press, Princeton, N.J, 1976.

[2] Alan Ross Anderson, Nuel D. Belnap Jr, and J. Michael Dunn. Entailment, Vol. II: The Logic of Relevance and Necessity. Princeton University Press, 2017.

[3] José Miguel Blanco. EF4, EF4-M and EF4-Ł: A companion to BN4 and two modal four-valued systems without strong Łukasiewicz-type modal paradoxes. Logic and Logical Philosophy, 31(1):75–104, 2022.

[4] José Miguel Blanco, Sandra M. López, and Marcos M. Recio. On how the 2 set-up Routley-Meyer semantics are a specific case of the reduced general Routley-Meyer semantics in the context of some 4-valued logics. Journal of Applied Logics - IFCOLOG Journal of Logics and their Applications, 10, 2023.

[10] These two works, together with [9, 3, 17] complete the research about the companions of BN4.

[5] Ross Brady. Universal Logic. CSLI Publications, 2006.

[6] Ross T. Brady. Completeness Proofs for the Systems RM3 and BN4. Logique et Analyse, 25(97):9–32, 1982. Publisher: Peeters Publishers.

[7] Ross T. Brady. Relevant Implication and the Case for a Weaker Logic: Dedicated to Robert K. Meyer on the occasion of his 60th birthday. Journal of Philosophical Logic, 25(2):151–183, 1996.

[8] Lou Goble. An Incomplete Relevant Modal Logic. Journal of Philosophical Logic, 29(1):103–119, 2000.

[9] Sandra M. López. Ternary relational semantics for the variants of BN4 and E4 which contain Routley and Meyer's logic B. The Bulletin of the Section of Logic, 51(1):27–56, 2022.

[10] Sandra M. López. Belnap-Dunn Semantics for the Variants of BN4 and E4 which Contain Routley and Meyer's Logic B. Logic and Logical Philosophy, 31(1):29–56, 2022. https://apcz.umk.pl/LLP/article/view/29336.

[11] Robert K. Meyer, Steve Giambrone, and Ross T. Brady. Where Gamma fails. Studia Logica, 43:247–256, 1984.

[12] Chris Mortensen. Model structures and set algebras for Sugihara matrices. Notre Dame Journal of Formal Logic, 23(1):85–90, 1982.

[13] Hitoshi Omori and Heinrich Wansing. 40 years of FDE: An introductory overview. Studia Logica, 105:1021–1049, 2017.

[14] Gemma Robles, José Miguel Blanco, Sandra M. López, Jesús R. Paradela, and Marcos M. Recio. Relational semantics for the 4-valued relevant logics BN4 and E4. Logic and Logical Philosophy, 25(2):173–201, 2016.

[15] Gemma Robles, Sandra M. López, José M. Blanco, Marcos M. Recio, and Jesús R. Paradela. A 2-set-up Routley-Meyer Semantics for the 4-valued Relevant Logic E4. Bulletin of the Section of Logic, 45(2):93–109, June 2016. Number: 2.

[16] Gemma Robles, Sandra M. López, and José Miguel Blanco. Relational semantics for the paraconsistent and paracomplete 4-valued logic PŁ4. Logic and Logical Philosophy, 2022.

[17] Gemma Robles and José Méndez. A companion to Brady's 4-valued relevant logic BN4: The 4-valued logic of entailment E4. Logic Journal of the IGLP, 24(5):838–858, 2016.

[18] Gemma Robles and José M. Méndez, editors. Routley-Meyer Ternary Relational Semantics for Intuitionistic-type Negations. Academic Press, 2018.

[19] Richard Routley and Robert K. Meyer. The semantics of entailment — III. Journal of Philosophical Logic, 1(2):192–208, 1972.

[20] Richard Routley and Robert K. Meyer. The semantics of entailment—II. Journal of Philosophical Logic, 1(1):53–73, 1972.

[21] Richard Routley and Robert K. Meyer. The Semantics of Entailment. In Hughes Leblanc, editor, Truth, Syntax, and Modality: Proceedings Of The Temple University Conference On Alternative Semantlcs, pages 199–243. Amsterdam: North-Holland Publishing Company, 1973.

[22] Richard Routley, Robert K. Meyer, Valerie Plumwood, and Ross T. Brady. Relevant Logics and Their Rivals, Vol. 1. Ridgeview, Atascadero, CA, 1982.

[23] John Slaney. Relevant Logic and Paraconsistency, pages 270–293. Springer Berlin Heidelberg, Berlin, Heidelberg, 2005.

[24] Natalya Tomova. A Lattice of Implicative Extensions of Regular Kleene's Logics. Reports on Mathematical Logic, 2012(47):173–182, 2012.