A note on the logic of Turing's halting paradox

Zach Weber*and Fernando Cano-Jorge†

Abstract

Projects directed at a *universal logic* (notably, Brady's [9]) have long struggled with paradoxes. In just the domain of computability, Hilbert's call for a general decision procedure was scuttled by Turing, using a diagonal argument. Indeed, Turing's halting problem can be straightforwardly viewed as a paradox, of the same type as others like the Liar and Curry's. This is confirmed here by reconstructing a halting predicate in a sequent calculus setting, thereby fitting a 'recipe' for paradox in [1]. The usual range of possible solutions then applies, including especially substructural approaches. Bringing this work as a response to Brady's "Starting the Dismantling of Classical Mathematics" [12], then, we assess his proposals to drop the law of excluded middle and other logical principles—asking whether this strategy avoids all versions of the halting paradox. Brady has well begun, in his idiom, the re-construction of mathematics; and there is more yet to do.

Dedication. Ross Brady has contributed a lifetime of insightful and careful research in logic, especially to the programmes of relevant logic and set theory. His results on the non-triviality of paraconsistent set theory remain among the most important landmarks in the field [7, 8, 15]. He has shown inspiring commitment to pursuing a philosophically and technically satisfactory answer to the deepest of questions. This paper is in grateful honor of his achievements.

1 Logic, Paradox, and Computability

In this note, we aim to get a better understanding of the logical structure of Turing's halting argument [38], cf. [12, p.293], and how to respond to it. We do this by drawing together some ongoing discussions in non-classical logic in a single setting, as a way of (1) clarifying the halting problem's place among other diagonal arguments; (2) engaging with efforts by Brady and others in a similar tradition to find a non-classical response to it; and so (3) opening up new paths forward in non-classical computability theory.

1.1 Universality via non-classicality

Part of logic is identifying patterns—formal, abstract similarities between arguments, whereby we can say that those fitting the 'shape' of e.g. modus ponens are valid. Insofar as Brady is right that "logic is about provability" [12, p.288], these patterns serve as a guide, regardless of content or context. The apotheosis of this exercise would be a *universal* logic, one that identifies patterns that apply in all times and places, and "to the exclusion of other ... logics" [9, p.x].

*Philosophy Programme, University of Otago. Email: zach.weber@otago.ac.nz

†Philosophy Programme, University of Otago, 95 Albany Street, Dunedin, 9054, New Zealand; Department of Philosophy, University of Canterbury, Private Bag 4800, Christchurch, 8140, New Zealand; Facultad de Filosofía, Universidad Panamericana, Augusto Rodin 498, Ciudad de México, 03920, México. E-mail: fernando.cano-jorge@otago.ac.nz ORCID: https://orcid.org/0000-0003-2835-6394

Universal logic projects infamously run into paradoxes, arguments which appear to fit patterns of valid reasoning, with true premises, but which seem to end in unacceptable or impossible conclusions [9, pp.45-50]; cf. [40, chs. 0, 1]. If we do not want to give up on the idea of universality, a bold response is to say that fewer of the canonized patterns are valid than classical logicians have thought. Here we find programs like Routley / Sylvan's [36], and maxims like "Gödel's first theorem would not be provable using a decent logic" [12, p.280, 296 (quoting Sylvan)]. Recently, Brady has suggested that his logic of *meaning containment* MC (or DJ, which is MC without distribution), which lacks ex contradictione quodlibet ECQ (and so is paraconsistent) and the law of excluded middle LEM (and so is paracomplete), fits the bill [10, 11, 13].

Logicians have tried hard to show that the paradoxes *themselves* fit into patterns. Past attempts to unify the paradoxes have been due to Russell (see [30, ch.9]), and Lawvere (see [43]), among others (see [40, ch.2]). It has turned out to be very difficult to find a pattern or schema that all the intuitively related paradoxes fit, without capturing too much, or conversely leaving something intuitively related out. One trend in this direction has been to turn to proof theory, to view paradoxical derivations in the context of a Gentzen-style sequent calculus; a recent attempt along these lines to provide a recipe for self-referential paradoxes is due to Ahmad [1].[1] If this sort of schematizing were successful, it would be another step closer to universality.

In [12] Brady works through a number of related diagonal arguments in the foundations of arithmetic and computability to prove MC's worth, arguing that without LEM no paradox results. Along with Cantor's diagonal argument and Gödel's first theorem, Brady includes Turing's halting problem. It seems to be widely accepted that the halting problem is a close relative of other diagonal incompleteness arguments.[2] If so, its classical impossibility could be overcome in the same way that other paradoxes are, via non-classical logic, recasting what is computable and not.

In this note we will make the connection between halting and the other paradoxes overt by presenting a two-place halting predicate that fits Ahmad's paradox schema, vindicating Brady's inclusion of it. We will also then emphasize where excluded middle really is (or is not) needed in a paradoxical derivation, which points in the direction of a substructural logic, not merely paraconsistent or paracomplete. Brady's work has taken us far, with farther still to go. We stand on the shoulders of giants—but with universality, we are reaching for the vault of the sky.

1.2 Paradoxes and non-classical solutions

Once it was clear, circa 1900, that some ancient paradoxes were not merely puzzling but rather dramatic blockages or even breakages in the foundations of logic itself, responses emerged. A venerable response was to claim that some form of arguments are invalid—and that they would be invalid even without the antinomies to prove it, e.g. as relevant logicians have long noticed, $A \to (B \to A)$ can't always be correct without making a nonsense of 'implies'. The idea that correcting our logic would also address the non-logical paradoxes is very appealing; cf. [36].

For a long time, people taking this approach seemed to think that we could mostly focus on

[1] Ahmad motivates his recipe in part with detailed criticism of previous attempts to schematize the paradoxes, especially the discussion in [39]; cf. [40, ch.2]. A key part of that debate is around the notion of *explanation*, providing a pattern that is not merely applicable but enlightening [3]. We take Ahmad's recipe to be a more neutral and schematic way to say when paradoxical arguments are structurally similar, without entering in to a debate about whether they are 'really' the same. E.g. the liar and the barber are the 'same' but one is a problem and the other is not [30, p.277].

[2] Although for the record, in Turing's original paper he says "what I shall prove is quite different from the well-known results of Gödel" [38, p.259].

dropping either one or both of ECQ or LEM, focusing our thinking on 'gluts' and 'gaps'. Most such approaches though have always struggled with Curry's paradox, e.g. [27].[3] This paradox seems to force abjuring principles that otherwise don't seem involved, like (operator) *contraction*, $(A \to (A \to B)) \to (A \to B)$. For while one can tell an epistemological story about the invalidity of the law of excluded middle, for instance, or its non-inclusion in a logic of meaning containment, it is harder to see why contraction should fail—and especially hard to walk away from its "deductive equivalent" $(A \wedge (A \to B)) \to B$, called pseudo-modus ponens. Nevertheless, walk away from contraction many, including Brady, did.[4]

This is where Brady's work really flourishes. Famously, he takes up the highly natural principles of naive set comprehension and extensionality, and shows how they can sit safely within a (weak) relevant logic, paradoxes or no. His systems can permit inconsistencies, provably non-trivial for the logic DKQ; but his considered view is that a consistent system based in the paracomplete logic DJQ, and its even weaker cousin MC, are philosophically preferable. Across a series of papers he shows for a variety of paradoxes that one can prove 'strange loops' of the form

$$B \leftrightarrow \neg B$$

but *not* outright contradictions

$$B \wedge \neg B$$

due to the absence of LEM.[5] (In intuitionistic logic, one has the scheme $A \to \neg A \vdash \neg A$, and this step is still valid, but Brady's systems lack this reductio principle; and arguably this principle seems to lose its motivation without LEM.) The role of LEM in diagonal arguments is observed in Brady and Rush [16] about Cantor's famous proofs.[6] Brady paints a picture where, without LEM, infinite sets call for reappraisal of Cantorian proofs; and, perhaps following intimations by Meyer [26, 25], a properly formulated arithmetic may call for reappraisal of Gödel's theorems. This connects to work like [37], suggesting that computability theory and undecidability results (§2.1 below) may be challenged from a non-classical point of view.

Circa the early part of this century, apparently new 'validity' versions of the Curry paradox surfaced [6]. While past non-classical solutions focus on logical operators, the new so-called 'validity curry' or 'V-curry' involves a (two-place) validity predicate $V(x, y)$ that seems only tractable on the substructural level—that is, calling in to question not just principles involving logical connectives, but structural rules or meta-inferences like cut and contraction[7]:

$$\frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,\text{Cut} \qquad\qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\text{CTR}$$

Calling these rules into question has led eventually to a great deal of work on substructural systems—which in turn, paradoxically, call in to question our very ability to characterize logic itself [4, 24].

[3] See [40, ch.4] for a summary. In [32], Priest points out that Curry's paradox and Löb's theorem deploy essentially the same argument; cf. [41].

[4] The justification for this can be linked on an independent basis to the *depth relevance* requirement. See [17, 23, 34].

[5] Priest contends that some paradoxes are still derivable without LEM [31, p.25], but this is a matter of (to our knowledge ongoing) debate with Brady ([14].

[6] In [16, ftnt 1], Brady notes discussions that led to this line of work.

[7] Here just given in single-conclusion forms; multiple-conclusion forms (with right-contraction, too) are used in §2.3 below.

In this context, it seems that certain principles and pieces of vocabulary—like a validity predicate—are not admissible on Brady's approach. This limitation is already recognized at length in uses of very weak relevant logics like B for truth predicates—that on such setups, due to extended Curry problems, validity is not definable [5, p.34]. While a few have notably maintained that V-paradoxes are not genuinely worrisome [20], these are the state of the art in terms of paradoxes any would-be universal non-classical logic needs to address.

As in Ahmad [1] (and citations therein), this literature tends to use Gentzen-style presentations of logics, using sequent calculi. In contrast, Brady's presentation of MC and related logics like DJ, as well as systems for inconsistent mathematics used by Routley, Meyer, and Mortensen, tend to be in Hilbert-style axiomatizations (though Brady does present a 'Gentzenization' of his systems in [9, ch.3§2, cf. p.110], following Dunn's similar treatment of relevant logic R in the 1970s). In this note, we will make our presentation in terms of a standard sequent calculus, with the aim of connecting the ideas in Brady's analysis to other literature on substructural treatment of paradoxes, bringing together two important lines of research.

Caveat: to be clear, we will not be claiming to have produced a problem for one system (Brady's axiomatic MC) by working in another system (a substructural Gentzen calculus), which would be a non-sequitur. We are, as the title suggests, rather looking for an abstract picture of the logic of Turing's argument, and how various proposals—like Brady's—fare, as a way forward for non-classical computability theories. We return to this in §3 below.

1.3 A recipe and a plan

Following Brady's inclusion of halting with the other diagonals, in §2.3 we show that the Halting Problem is plausibly of the same sort as other paradoxes. This is by making it fit a 'recipe' abstracted from paradoxes like the Liar and Curry by Ahmad. In Ahmad's analysis, the ingredients for paradox are [1, p.267-8]:

- (Transparency). A predicate P that is transparent in the sense of obeying two rules:

$$\frac{\vdash A}{\vdash P(\ulcorner A \urcorner)} \qquad \frac{}{P(\ulcorner A \urcorner) \vdash A}$$

- (Negative Modality). An operator m that obeys the rule:

$$\frac{A \vdash}{\vdash mA}$$

  where predicate P occurs in A.

- (Fixpoint). A sentence S that contains 'P' and is equivalent to mS.

All the expected paradoxes (arguably) arise from this recipe. In §3 we consider the implications for Brady's project. Let us turn to halting.

2 The halting paradox

2.1 The problem revisited

Turing's celebrated paper [38] is nowadays generally known as the genesis of modern computers and computability theory. His main motivation, surprisingly perhaps, was not to accomplish the technological revolution that came after this publication. In fact, he was trying to solve a logical problem haunting the foundations of mathematics: Hilbert's *Entscheidungsproblem* (decision problem), which, generally speaking, asks whether there is a general procedure (nowadays called an algorithm) that determines if some statement A is a theorem, i.e if A can be proved from a given set of axioms (say, of arithmetic or geometry) through logical rules. Alas, Turing's arguments show that no such general procedure exists. This, together with Gödel's incompleteness theorems, was the end of Hilbert's programme. Or so the story goes.[8]

In order to answer the *Entscheidungsproblem*, Turing had to come up with a mathematically precise formulation of the concepts involved in Hilbert's question, i.e. what it means to have some regulated step-by-step calculating procedure that, after a finite amount of time, produces a determinate result. This was the *raison d'être* of *Turing machines*. And once such a technical framework was available, it was easy to show that there is no general procedure that determines whether or not some statement is a theorem.

Turing's ingenious way around the *Entscheidungsproblem* was to ask an equivalent question he could phrase in terms of the computing machines framework he had designed: given that some Turing machines halt and produce a definite output while some others loop forever, is there a universal Turing machine that, given any other machine running on some fixed input, will determine if that machine will halt or not? This question came to be known as the *halting problem* and Turing answered it in the negative by showing that a contradiction follows if we assume such a universal Turing machine exists. This in turn showed, by equivalence, that the *Entscheidungsproblem* had a negative answer.

Both Turing and Church independently came to the same result. Computability theory has ever since taught its students that there are limits to what one can compute. And this is indeed so if we all play by the rules of classical logic. After almost a century since the results of Church and Turing, non-classical logical research has bloomed spectacularly. Non-classical foundations of mathematics are now something that logicians and mathematicians can explore in order to find where the limits of classical foundations blur. So we may today by all means ask: what does computability theory look like when resting on inconsistency-tolerant logical foundations? The first step is reconsidering the linchpin argument about uncomputability.

2.2 The argument

There are many ways to present the halting problem and answer it in the negative through *reductio* arguments. Let us consider the following version of the argument, one of the simplest ways to establish the negative result.

Suppose that we have a *halting program* $\mathfrak{halt}$ such that, for any program p and any input i,

$$\mathfrak{halt}(p, i) = \begin{cases} 1 & \text{if } p(i) \downarrow \\ 0 & \text{if } p(i) \uparrow \end{cases}$$

Here 1 and 0 mean "yes" and "no", respectively. Intuitively, $\mathfrak{halt}(p, i) = 1$ means "program p halts on input i" and $\mathfrak{halt}(p, i) = 0$ means "program p loops (does not halt) on input i". Following

[8] The reader is cautioned that the argument given in this section and below is offered as a rational reconstruction of a famous, almost folklore, proof (as found also in [12, p.293-294]). We do not claim to be doing textual historical scholarship on Turing's original paper. For that, see [21].

computability theory conventional notation, $p(i)\downarrow$ means that the (partial) function p is defined on the particular input i, i.e. that i is an element of the domain of p and thus $p(i)$ is the unique output obtained when p is applied to i; naturally, $p(i)\uparrow$ means the contrary, i.e. that p is undefined on input i, so i is not an element of the domain of p and hence applying p to i does not produce an output.[9]

We can use $\mathfrak{halt}$ to construct a new "self-halt" program that answers positively when a given program p will halt when we feed (some encoding of) itself as an input:

$$\mathfrak{sh}(p) = \begin{cases} 1 & \text{if } \mathfrak{halt}(p,p) = 1 \\ \uparrow & \text{if } \mathfrak{halt}(p,p) = 0 \end{cases}$$

And, of course, we can also build the opposite "self-loop" program:

$$\mathfrak{sl}(p) = \begin{cases} 1 & \text{if } \mathfrak{sh}(p)\uparrow \\ \uparrow & \text{if } \mathfrak{sh}(p) = 1 \end{cases}$$

Here comes the paradox: does $\mathfrak{sl}(\mathfrak{sl})$ halt or loop? If $\mathfrak{sl}(\mathfrak{sl})\downarrow$ then $\mathfrak{halt}(\mathfrak{sl},\mathfrak{sl}) = 1$, so $\mathfrak{sh}(\mathfrak{sl}) = 1$, whence $\mathfrak{sl}(\mathfrak{sl})\uparrow$. On the other hand, if $\mathfrak{sl}(\mathfrak{sl})\uparrow$ then $\mathfrak{halt}(\mathfrak{sl},\mathfrak{sl}) = 0$, whence $\mathfrak{sh}(\mathfrak{sl})\uparrow$, which in turn implies that $\mathfrak{sl}(\mathfrak{sl}) = 1$, so it is not the case that $\mathfrak{sl}(\mathfrak{sl})\uparrow$, i.e. $\mathfrak{sl}(\mathfrak{sl})\downarrow$. Therefore $\mathfrak{sl}(\mathfrak{sl})\downarrow$ iff $\mathfrak{sl}(\mathfrak{sl})\uparrow$, which by definition is $\mathfrak{sl} \in \mathrm{dom}(\mathfrak{sl})$ iff $\mathfrak{sl} \notin \mathrm{dom}(\mathfrak{sl})$, from which the contradiction $\mathfrak{sl} \in \mathrm{dom}(\mathfrak{sl})$ and $\mathfrak{sl} \notin \mathrm{dom}(\mathfrak{sl})$ follows.[10]

Since our only assumption was the existence of $\mathfrak{halt}$ and it led to a contradiction, classical reasoning will have us conclude that a halting program cannot exist, on pain of triviality. This is certainly so if one is committed to classical logic as the correct foundations of mathematics and, in particular, of computability theory. Let us now look at the halting paradox more carefully, logically speaking, so that we can isolate all its moving parts and consider ways to deal with it.

2.3 From halting to triviality

We give the halting paradox following the recipe from [1]. We work in a classical multiple conclusion sequent calculus, to see what principles are involved, without endorsing these principles or attributing them to Brady. This include the left and right negation rules

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\neg\,\mathrm{L} \qquad\qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\neg\,\mathrm{R}$$

as well as the structural rules of contraction (CTR) and transitivity (Cut). We also assume Exchange on the left and right, so Γ, Δ is the same as Δ, Γ.

With this negation satisfying Ahmad's 'negative modality' (§1.3), what is needed is a halting predicate that is transparent, and gives rise to a fixed point for negation. We define a two-place predicate H that applies to natural numbers. The first parameter is a number encoding the instructions to compute a function, and the second parameter is the number that the encoded function takes as input. So if f is a function on the natural numbers, $\ulcorner f \urcorner$ is the code of the instructions to compute f. One may use Turing's *description numbers* for this purpose: given

[9]It is common to read $f(x)\downarrow$ as "program f halts on input x" and $f(x)\uparrow$ as "program f loops on input x".

[10]Note again that $A \leftrightarrow \neg A \dashv\vdash A \wedge \neg A$ is valid in classical and intuitionistic logic [12, p.297, footnote 4]. This is a key moving part in Brady's analysis.

some Turing machine M that computes f, $\ulcorner f \urcorner$ is the description number obtained by reducing each m-configuration in the table for M to some sequence of numbers according to the process described in [38, §5]. Predicate H is to be understood as saying that x applied to y halts, or $\mathfrak{halt}(\ulcorner x \urcorner, \ulcorner y \urcorner) = 1$.

The informal considerations in §2.2 lead to the following.

$$\frac{}{H(\ulcorner x \urcorner, \ulcorner y \urcorner) \dashv\vdash x(\ulcorner y \urcorner)\downarrow}\text{H1} \qquad \frac{}{\neg H(\ulcorner x \urcorner, \ulcorner y \urcorner) \dashv\vdash x(\ulcorner y \urcorner)\uparrow}\text{H2}$$

$$\frac{\Gamma,\, x(\ulcorner x \urcorner)\downarrow \vdash \Delta}{\Gamma,\, sh(\ulcorner x \urcorner) = 1 \vdash \Delta}\text{SH1L} \qquad \frac{\Gamma \vdash x(\ulcorner x \urcorner)\downarrow,\, \Delta}{\Gamma \vdash sh(\ulcorner x \urcorner) = 1,\, \Delta}\text{SH1R}$$

$$\frac{\Gamma,\, x(\ulcorner x \urcorner)\uparrow \vdash \Delta}{\Gamma,\, sh(\ulcorner x \urcorner)\uparrow \vdash \Delta}\text{SH2L} \qquad \frac{\Gamma \vdash x(\ulcorner x \urcorner)\uparrow,\, \Delta}{\Gamma \vdash sh(\ulcorner x \urcorner)\uparrow,\, \Delta}\text{SH2R}$$

$$\frac{\Gamma,\, sh(\ulcorner x \urcorner) = 1 \vdash \Delta}{\Gamma,\, sl(\ulcorner x \urcorner)\uparrow \vdash \Delta}\text{SL1L} \qquad \frac{\Gamma \vdash sh(\ulcorner x \urcorner) = 1,\, \Delta}{\Gamma \vdash sl(\ulcorner x \urcorner)\uparrow,\, \Delta}\text{SL1R}$$

$$\frac{\Gamma,\, sh(\ulcorner x \urcorner)\uparrow \vdash \Delta}{\Gamma,\, sl(\ulcorner x \urcorner) = 1 \vdash \Delta}\text{SL2L} \qquad \frac{\Gamma \vdash sh(\ulcorner x \urcorner)\uparrow,\, \Delta}{\Gamma \vdash sl(\ulcorner x \urcorner) = 1,\, \Delta}\text{SL2R}$$

Auxiliary rules:

$$\frac{\Gamma,\, x(\ulcorner y \urcorner) = 1 \vdash \Delta}{\Gamma,\, x(\ulcorner y \urcorner)\downarrow \vdash \Delta}\text{DL} \qquad \frac{\Gamma \vdash x(\ulcorner y \urcorner) = 1,\, \Delta}{\Gamma \vdash x(\ulcorner y \urcorner)\downarrow,\, \Delta}\text{DR}$$

The only negation rule used with up/down arrows is:

$$\frac{\Gamma, \neg x(\ulcorner y \urcorner)\downarrow \vdash \Delta}{\Gamma, x(\ulcorner y \urcorner)\uparrow \vdash \Delta}\text{L}\neg\downarrow$$

The initial sequents H1 and H2 are really two axioms, one for each direction of $\vdash$ and $\dashv$. Note that we could replace them with four rules, introducing on the right and left, e.g.

$$\frac{\Gamma\ \vdash \neg H(x,y), \Delta}{\Gamma\ \vdash x(y)\uparrow, \Delta}$$

Then all the proofs could proceed only from initial identity sequents. We present things this way, first, because this emphasizes the role of assumptions about a putative halting function in the original reductio-style arguments; and second, going this way also makes some cuts explicit in the derivations below.

Some of these rules may be derivable from others—for example, H2 would follow from H1, using more negation rules—but we list them all for the sake of explicitness, and on the grounds that they can be justified by the informal considerations in §2.2.[11] If desired, one could replace DL and DR with a more general pair of one-way inference rules from $f(a) = b$ to $f(a)\downarrow$ for any function f and any natural numbers a, b.

All this in place, then, $H(x, y)$ is our transparent predicate, and the *Turing sentence* $H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)$ is the problematic fixpoint: $H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)$ iff $\neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)$.

Let Δ_1 be the following derivation:[12]

[11] As with the informal argument in §2.2, strictly speaking sh is not needed, since sl can be defined directly in terms of H. (Thanks to a referee for pointing this out.) However, the purpose here is to provide a rational reconstruction of a *paradox*—an apparently acceptable argument that has apparently unacceptable consequences. Skipping out the steps through sh and cutting directly to the contradiction through sl loses the intuitive pull of the (informal) argument. Analagously, one could formulate a sorites paradox with a premise like "If two grains of sand is not a heap, then 10,000 grains of sand is not a heap", i.e. by making the jumps quite large, but then there is no sense of a 'forced march' and no sense of needing to explain what went wrong. The argument retains its plausibility only through incremental changes. Intermediary steps are an integral part of the paradoxicality. See [40, ch.1§2].

[12] We are using Δ for sets of formulas, and also subscripted Δ_is for names of derivations. Context will disambiguate.

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\dfrac{\dfrac{}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash sl(\ulcorner sl \urcorner)\downarrow}\,\text{H1}}{\vdash sl(\ulcorner sl \urcorner)\downarrow, \neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}\,\neg\text{R}
\qquad
\dfrac{}{\neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash sl(\ulcorner sl \urcorner)\uparrow}\,\text{H2}
}{\vdash sl(\ulcorner sl \urcorner)\downarrow,\, sl(\ulcorner sl \urcorner)\uparrow}\,\text{Cut}
}{\vdash sl(\ulcorner sl \urcorner)\downarrow,\, sh(\ulcorner sl \urcorner)\uparrow}\,\text{SH2R}
}{\vdash sl(\ulcorner sl \urcorner)\downarrow,\, sl(\ulcorner sl \urcorner) = 1}\,\text{SL2R}
}{\vdash sl(\ulcorner sl \urcorner)\downarrow,\, sl(\ulcorner sl \urcorner)\downarrow}\,\text{DR}
}{\vdash sl(\ulcorner sl \urcorner)\downarrow}\,\text{CTR}
\qquad
\dfrac{}{sl(\ulcorner sl \urcorner)\downarrow \vdash H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}\,\text{H1}
}{\vdash H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}\,\text{Cut}
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Let Δ_2 be the following derivation:

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\dfrac{}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash sl(\ulcorner sl \urcorner)\downarrow}\,\text{H1}
}{\neg sl(\ulcorner sl \urcorner)\downarrow,\, H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash}\,\neg\text{L}
}{sl(\ulcorner sl \urcorner)\uparrow,\, H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash}\,\text{L}\neg\downarrow
}{sh(\ulcorner sl \urcorner)\uparrow,\, H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash}\,\text{SH2L}
}{sl(\ulcorner sl \urcorner) = 1,\, H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash}\,\text{SL2L}
}{sl(\ulcorner sl \urcorner)\downarrow,\, H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash}\,\text{DL}
\qquad
\dfrac{}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash sl(\ulcorner sl \urcorner)\downarrow}\,\text{H1}
}{\dfrac{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner),\, H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash}\,\text{CTR}}\,\text{Cut}
$$

Then we can derive the empty sequent:

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\dfrac{\Delta_1}{\vdash H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}
\qquad
\dfrac{\Delta_2}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash}
}{\vdash}\,\text{Cut}
$$

Triviality.

The derivation of triviality does not overtly use LEM. If so, Brady's denial of the LEM alone does not immediately help here; but the negation rules do seem undercut without the LEM.[13] More fundamentally, Brady's rejection of the earlier rule of explosion ($\neg$-left) and use of a de Morgan negation does the work in preventing collapse. This proof uses contraction and cut, but also the negation rules before either of those.

2.4 From halting to inconsistency

The following is a derivation showing the Turing sentence is contradictory.

Let Δ_3 be the following derivation:

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\dfrac{}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash sl(\ulcorner sl \urcorner)\downarrow}\,\text{H1}
}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash sh(\ulcorner sl \urcorner) = 1}\,\text{SH1R}
}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash sl(\ulcorner sl \urcorner)\uparrow}\,\text{SL1R}
\qquad
\dfrac{}{sl(\ulcorner sl \urcorner)\uparrow \vdash \neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}\,\text{H2}
}{\dfrac{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash \neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}{\vdash H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \to \neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}\,\to\text{R}}\,\text{Cut}
$$

Let Δ_4 be the following derivation:

[13] In [29], an unpublished note where Petersen explores the halting problem using his contraction-free logic, he assumes $\neg\uparrow$ and $\neg\downarrow$ rules and also asks "does *tertium non datur* hold for 'defined'?, i.e. $\{f\}(f)\downarrow \vee \{f\}(f)\uparrow$?". His thought is that if LEM holds in such a way, he may be able to replace occurrences of contraction in the proof of the unsolvability of the halting problem. However, he seems to conclude that no contraction-free proof of the diagonal argument is available. See §3.5 below.

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\dfrac{
\dfrac{}{\neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash sl(\ulcorner sl \urcorner)\uparrow}\;\text{H2}
}{\neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash sh(\ulcorner sl \urcorner)\uparrow}\;\text{SH2R}
}{\neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash sl(\ulcorner sl \urcorner) = 1}\;\text{SL2R}
}{\neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash sl(\ulcorner sl \urcorner)\downarrow}\;\text{DR}
\qquad
\dfrac{}{sl(\ulcorner sl \urcorner)\downarrow \vdash H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}\;\text{H1}
}{
\dfrac{\neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}{\vdash \neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \to H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}\;\to\text{R}
}\;\text{Cut}
$$

Then:

$$
\dfrac{
\dfrac{
\dfrac{\Delta_3}{\vdash H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \to \neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}
\qquad
\dfrac{\Delta_4}{\vdash \neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \to H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}
}{\vdash H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \leftrightarrow \neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}
}{\vdash H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \wedge \neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}
$$

Contradiction.

These derivations do not use the $\neg$-left or right rules. A $\to$ connective based on meaning containment may not introduce as shown in these proofs (see §3) but we could stop Δ_3 and Δ_4 one line earlier and still get the two way problematic $H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \dashv\vdash \neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)$.

Thus these arguments could go through in Brady's approach—that is, up to the last step, which is not taken by Brady: we do not go from $A \leftrightarrow \neg A$ to $A \wedge \neg A$, because of the absence of the LEM. So Brady has a solution at the last step to the inconsistent version of the paradox.

2.5 Triviality without negation?

Is there a de Morgan negation-free paradox derivation? This is where extended Curry-like paradoxes have really shown their teeth. There is such a problem, if there is a two-place operator or predicate C such that if $\vdash C(A, B)$ then $A \vdash B$—which amounts to a way of moving A from right to left. One clear candidate for such an operator is intuitionistic-like negation $A \to 0 = 1$, assuming that in Brady's systems we have

$$
\dfrac{\Gamma \vdash A \to B, \Delta}{\Gamma, A \vdash B, \Delta}
$$

This leads to possible trouble as follows.

Informal considerations suggest a modified rule (two way, in keeping with rules H1 and H2):

$$
\dfrac{}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \dashv\vdash sh(\ulcorner sl \urcorner)\uparrow \to 0 = 1}\;\text{HSH}
$$

We can argue for this rule as follows:

- Left-to-right, suppose $H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)$. That is, $\mathfrak{halt}(\ulcorner sl \urcorner, \ulcorner sl \urcorner) = 1$. If $sh(\ulcorner sl \urcorner)\uparrow$ then $sl(\ulcorner sl \urcorner) = 1$, and then $sl(\ulcorner sl \urcorner)\downarrow$, and then $sh(\ulcorner sl \urcorner) = 1$, and then $sl(\ulcorner sl \urcorner)\uparrow$, which by definition implies $\mathfrak{halt}(\ulcorner sl \urcorner, \ulcorner sl \urcorner) = 0$; whence $0 = 1$ on hypothesis.

- For the right-to-left direction, this is motivated by the reductio rule, $(A \to 0 = 1) \vdash \neg A$, in the following argument. (This argument instance could still be accepted without a general commitment to the reductio rule or LEM; see §3.5 below.) Suppose $sh(\ulcorner sl \urcorner)\uparrow\to 0 = 1$. Since $sl(\ulcorner sl \urcorner)\uparrow$ implies $sh(\ulcorner sl \urcorner)\uparrow$ (by SH2R), it follows that $sl(\ulcorner sl \urcorner)\uparrow\to 0 = 1$, which means $\neg sl(\ulcorner sl \urcorner)\uparrow$, i.e. $sl(\ulcorner sl \urcorner)\downarrow$, whence $H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)$.

With this rule, then let Δ_5 be the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash sh(\ulcorner sl \urcorner) \uparrow \to 0 = 1}\ \text{HSH}}{sh(\ulcorner sl \urcorner) \uparrow, H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash 0 = 1}\ {\to}\text{L}}{sl(\ulcorner sl \urcorner) = 1, H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash 0 = 1}\ \text{SL2L}}{sl(\ulcorner sl \urcorner) \downarrow, H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash 0 = 1}\ \text{DL} \qquad \dfrac{}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash sl(\ulcorner sl \urcorner) \downarrow}\ \text{H1}}{\dfrac{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner), H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash 0 = 1}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash 0 = 1}\ \text{CTR}}\ \text{Cut}$$

Now, we recall that Δ_1 proved $\vdash H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)$; so we could put these together for

$$\dfrac{\dfrac{\Delta_1}{\vdash H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)} \qquad \dfrac{\Delta_5}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash 0 = 1}}{\vdash 0 = 1}\ \text{Cut}$$

But since Δ_1 uses negation, the conclusion is avoided, and this isn't a worry for Brady.

The Turing sentence can be derived without any (de Morgan) negation, however. Here is Δ_6:

$$\dfrac{\dfrac{\dfrac{}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash sh(\ulcorner sl \urcorner) \uparrow \to 0 = 1}\ \text{HSH}}{sh(\ulcorner sl \urcorner) \uparrow, H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash 0 = 1}\ {\to}\text{L} \qquad \dfrac{}{sl(\ulcorner sl \urcorner) \downarrow \vdash H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}\ \text{H1}}{sh(\ulcorner sl \urcorner) \uparrow, sl(\ulcorner sl \urcorner) \downarrow \vdash 0 = 1}\ \text{Cut}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{sh(\ulcorner sl \urcorner) \uparrow, sl(\ulcorner sl \urcorner) \downarrow \vdash 0 = 1}{sh(\ulcorner sl \urcorner) \uparrow, sh(\ulcorner sl \urcorner) = 1 \ \vdash 0 = 1}\ \text{SH1L}}{sh(\ulcorner sl \urcorner) \uparrow, sl(\ulcorner sl \urcorner) \uparrow \vdash 0 = 1}\ \text{SL1L}}{sh(\ulcorner sl \urcorner) \uparrow, sh(\ulcorner sl \urcorner) \uparrow \vdash 0 = 1}\ \text{SH2L}}{sh(\ulcorner sl \urcorner) \uparrow \vdash 0 = 1}\ \text{CTR}}{\vdash sh(\ulcorner sl \urcorner) \uparrow \to 0 = 1}\ {\to}\text{R}}{\vdash H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}\ \text{HSH and Cut}$$

Cut this together with Δ_5 to get $0 = 1$. There are similar alternative proofs, e.g. using $\to$-introduction twice and then operator contraction. Since operator contraction went off the table in §1.2, these are left to the curious reader.

Now it seems we have a problematic derivation that does not use $\to$-contraction, or LEM. Using only arrow rules and structural contraction and cut we seem to prove $0 = 1$. This is not quite the empty sequent, but it does not seem good. In Brady's arithmetic in MC [10], which is consistent, $0 = 1$ is not on.

Why not? Some researchers in paraconsistent computability theory seem unsure, e.g. [37]; and indeed there may be some (slim) room for considering the possibility that $0 = 1$ is at least tolerable, depending on the underlying paraconsistent arithmetic.[14] But for Brady, the non-triviality of naive set theory is established by showing that some 'absurd' sentences, like $\forall x \forall y (x \in y)$, are not satisfied in a model of the theory. In Sylvan/Brady set theory, if 0 is defined in terms of such an absurd sentence, and 1 is defined as $\{0\}$, then *proving* $0 = \{0\}$ must be ruled out to maintain coherence; cf. [42].

Brady's way out here can be denial of the $\to$ intro rule used at the end of Δ_6. This is related to the denial of operator contraction already, and the associated 'pseudo' modus ponens: $\to$ elimination is a valid rule, but not a valid sentence-level axiom. We turn now to further analysis on these derivations.

[14] A related functionality problem for inconsistent computability theory is raised in [18].

3 Options, observations, and open questions

Having studied some derivations of halting, in conversation with Brady's ideas, let us consider a few questions this raises.

3.1 Options

The halting paradox follows the same pattern as other paradoxes.[15] There are multiple ways to address the halting paradox—and, more generally, the paradoxes covered by Ahmad's recipe.

1. Restricting or abandoning Cut, as in [35].
2. Restricting or abandoning (structural) Contraction, as in [40] and [2].
3. Restricting or abandoning ¬L and ¬R.

Solutions 1 and 2 belong to the substructural approach to paradoxes. Solution 1 avoids uses of cut associated with H1 and H2, and the last step in the derivation in §2.3. Brady's logics are trending toward Solution 2: they do not include operator contraction.

Dropping cut, one can maintain all the classical rules for logical operators, and avoid debate about whether to restrict e.g. negation. Keeping cut, we've seen that if enough care is taken with connectives—restricting negation rules, denying →-introduction unrestrictedly—then it is possible to defuse the derivations given above.

3.2 Multiple vs single conclusions

One thing to clear away is a question of how much depends on the particular presentation of a logic, e.g. a single versus a multiple conclusion Gentzen approach. Given a fixed point $B \dashv\vdash \neg B$, the latter comes to grief very quickly:

$$\dfrac{\dfrac{\dfrac{B \vdash \neg B}{\vdash \neg B, \neg B}}{\vdash \neg B} \qquad \dfrac{\dfrac{\neg B \vdash B}{\neg B, \neg B \vdash}}{\neg B \vdash}}{\vdash}$$

An opponent of exhaustion and exclusion would deny both sides of this. But we can give a derivation that does not require multiple conclusions:

$$\text{Cut}\;\dfrac{\text{H1 and Cut}\;\dfrac{\text{DR}\;\dfrac{\text{SL2R}\;\dfrac{\text{SH2R}\;\dfrac{\text{H2 and Cut}\;\dfrac{\neg\text{R}\;\dfrac{\dfrac{\Delta_2}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash}}{\vdash \neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)}}{\vdash sl(\ulcorner sl \urcorner)\uparrow}}{\vdash sh(\ulcorner sl \urcorner)\uparrow}}{\vdash sl(\ulcorner sl \urcorner) = 1}}{\vdash sl(\ulcorner sl \urcorner)\downarrow}}{\vdash H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)} \qquad \dfrac{\Delta_2}{H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \vdash}}{\vdash}$$

[15] In [43], it is shown that the existence of non-recursively enumerable languages follows from Lawvere's paradox schema, as does the unsolvability of the halting problem, and many other paradoxes. Other important results from computability theory fall into this pattern, like the fact that there is an oracle B such that the class of all problems that can be solved with deterministic oracle Turing machines in polynomial time consulting B is different from the class of all problems that can be solved with nondeterministic oracle Turing machines in polynomial time consulting B. Thanks to Luis Estrada-González for pointing this out.

Here there is no contraction at any step (though we have the same *proof* used twice[16]). So similar issues arise in single-conclusion settings.

3.3 Negation and contraction

Turning from this to the issues of LEM and contraction, what is the relationship between Solution 2 and Solution 3? One may argue that, if we take an non-contractive substructural approach, this *implies* a paranormal (i.e. non-exclusive, non-exhaustive) approach. This is due to contraction being used in the following derivation of the LEM:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A}}{A \vdash A \lor \neg A}\,\lor\mathrm{R}}{\vdash \neg A, A \lor \neg A}\,\neg\mathrm{R}}{\vdash A \lor \neg A, A \lor \neg A}\,\lor\mathrm{R}}{\vdash A \lor \neg A}\,\mathrm{CTR}$$

The last step is by contraction; see [28, p.25], where this derivation is given to illustrate the "intuitionistic objection" as a reason for dropping contraction.[17] Without contraction, there is no commitment to LEM unless it is there already. One can then further argue that Brady is right: without LEM, there are no outright contradictions of the form $A \land \neg A$, just circular anomalies of the form $A \leftrightarrow \neg A$.

If one is aiming for paraconsistency (as Brady and other 'universalists' in this tradition do), and keeps the Cut rule, and one admits pieces of vocabulary like a halting predicate, then restricting contraction is not enough. In particular, if we expect inconsistent information along the lines of §2.4, it seems to us the negation-left rule (from $\vdash A$ to $\neg A \vdash$) is very much like ECQ and to be dropped. In [9, p.134] Brady indeed only adopts de Morgan principles for the Gentenization of his logic. Logics in this vicinity are well-prepared for paradox—as long as the paradoxical predicates do not covertly reintroduce some banished behaviour.

3.4 Conditionals

Indeed, we may ask: how much depends on a conditional operator $\to$ and its rules, versus focusing on structural properties of $\vdash$? On Brady's approach, modus ponens is a valid *rule*, but is deductively equivalent to contraction if stated as an *axiom*—hence $(A \land (A \to B)) \to B$ is 'pseudo' modus ponens. Some apparently true things, like about validity, cannot be expressed within the system (including some about the system itself). Extended validity predicates would be inadmissible for logics like MC.

For the record, an alternative proposal is something like Priest's LP, where modus ponens (for the material conditional) can be stated as an axiom, but is not a valid rule: one may have $\vdash A \supset B$ without $A \vdash B$. So, much as a paranormal approach (with cut) cannot have both $\neg$-L and $\neg$-R rules, similarly an approach with structural contraction seems unable to have both conditional-introduction and conditional-elimination rules unrestrictedly.

Again, as in §2.5, the issue really seems to be some C such that $\vdash C(A, B)$ iff $A \vdash B$. Abstractly, the form of a dangerously paradoxical object $\mathfrak{G}$ is not in terms of connectives at all, but in moving

[16]This extends the question of 'contracting on theorems'—which seems acceptable, if subtle [40, p.142]—to contracting on whole proofs. The issue warrants further study for serious non-contractivists.

[17]Also note that the disjunction rules used in the derivation of LEM are not Ketonen-style, otherwise contraction would not be used. See [28, pp.11-14].

back and forth across the turnstile unchanged: $\vdash \mathfrak{G}$ iff $\mathfrak{G} \vdash$. This suggests—as many have suggested [19, ch.2, cf. p.207-8]—that analysis at the level of negation and even implication is not the full story and calls for substructural work.

3.5 $0 = 1$ and the addition of classical sentences

Similarly, and as hinted in §2.5, a problem not addressed by any would-be paraconsistent-cum-universal system is what lies beneath the negation inconsistency of halting. Suppose we accept something like $H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \leftrightarrow \neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)$ or even $H(\ulcorner sl \urcorner, \ulcorner sl \urcorner) \wedge \neg H(\ulcorner sl \urcorner, \ulcorner sl \urcorner)$. What does it mean? It means that there is some g such that

$$\mathfrak{halt}(g, g) = 1 \leftrightarrow \mathfrak{halt}(g, g) = 0$$

or even

$$\mathfrak{halt}(g, g) = 1 \wedge \mathfrak{halt}(g, g) = 0$$

The latter would imply the (apparently catastrophic) $0 = 1$, assuming transitivity of $=$. This might seem like a point in Brady's favor: drop LEM. But then, $\mathfrak{halt}$ does not even seem well-defined to begin with, since it presumes in the background that there are only two possible cases to consider.[18]

Now, most people who reject the general validity of LEM still accept that many instances of it are correct; it is just not *always* correct. Brady endorses the addition of what he calls 'classical' sentences that do satisfy instances of LEM [9, §1.7, p.42]. Could the sentences involved in defining and reasoning about halting be classical in this way? It would be question begging to say no *because* otherwise one reaches inconsistency. What Brady says is (where his $D(D)$ is our diagonal $sl(\ulcorner sl \urcorner)$):

> LEM should be supported through one of its disjuncts, $D(D)$ halts or $D(D)$ moves the cursor to the right. Neither of these have the required support as they would both need some recursive deductive process for the argument to their respective conclusions to take shape and this has not been established, especially in light of the classical undecidability results [12, p.294].

It is a question for further discussion whether this is enough justification for failure of LEM, or whether it falls back on an assumption of classical *reductio ad contradictionem* as definitive. Would the special cases of some 'negation' rules for functions over the natural numbers, like

$$\frac{\Gamma \vdash x(y)\downarrow, \Delta}{\Gamma, x(y)\uparrow \vdash \Delta} \qquad \frac{\Gamma, x(y)\downarrow \vdash \Delta}{\Gamma \vdash x(y)\uparrow, \Delta}$$

$$\frac{\Gamma \vdash x(y)\uparrow, \Delta}{\Gamma, x(y)\downarrow \vdash \Delta} \qquad \frac{\Gamma, x(y)\uparrow \vdash \Delta}{\Gamma \vdash x(y)\downarrow, \Delta}$$

be reasonable, even if LEM is not? (Cf. footnote 13 above about [29].) With these, a completely connective-free derivation of triviality is possible, using only structural rules (cf. §3.4 above). Even more specifically, if there were a plausible recursive deductive process that made LEM apply to halting (as is argued by [37]), it would return the question posed in §2.4: what does it mean to halt and not halt? The prospects for an inconsistent approach remain mostly undeveloped. Time will tell.

[18] Cf. discussion of the role of classicality in the meaning of functions and their calculability in [22, p.5].

3.6 Simple and Absolute Consistency

Finally, we can ask a few questions about the shape of Brady's preferred system.

With respect to paradoxes, the upshot is to accept 'loops' of the form $B \leftrightarrow \neg B$ but not inconsistencies of the form $B \wedge \neg B$, whence Brady can prove the simple consistency of his systems. One question, then, is: if there are in fact no contradictions in the offing, why is the logic also paraconsistent at all? Or put another way, if absolute consistency (non-triviality) is already available, what is gained by avoiding contradiction?

Brady could reply that paraconsistency is there just as a result of logical relevance. Indeed, on relevantist grounds, inferring an arbitrary proposition from a contradiction is bad reasoning due to the lack of content being shared between antecedent and consequent. Thus, the failure of ECQ need not mean that the logic will encounter contradictory theorems from which non-trivial consequences may follow, but only that ECQ cannot be valid since the entailment and implication relations are relevant. That these features are key when it comes to dissolving paradoxes and avoiding limitative results was already advanced by Routley in [36], who, unlike Brady, allowed contradictions in e.g. set theory.

A second question concerns the meaning of the loops we are meant to accept so sanguinely. While not outright inconsistent, they are still odd, and point to the underlying motivations for this project. If we seek to reconstruct arithmetic and computability theory on this basis, but are somehow freed of Turing's halting 'paradox', one may worry that a looping proto-inconsistency that never halts is not a significant change from the classical situation.

4 Conclusion

Turing's halting paradox fits a familiar diagonal pattern, calling attention to logical principles involving negation and contraction. As ever, a universalist—especially one intending to develop a non-standard computability theory—looks for what principles are essential to problematic derivations. Dropping a principle that is used in some derivations but not others is only a very partial solution, at best. Dropping a principle that is used in all derivations but also is needed for apparently innocuous arguments that cannot be made without it is an overly-total solution, at worst. Balance is needed.

A universal logic, if one exists, seems to require being weak. Brady has long included 'meta-rules' (following Meyer) in his axiomatizations [12, p.282], giving a kind of hybrid Hilbert system with axioms, rules, and what could be structural rules. Making some alterations to the structural rules seems to us still within the constraints and motivation for a logic of meaning containment. Whether halting is more like the liar (treatable with de Morgan negation) or Curry (treatable with more drastic constraints on contraction)—or whether the distinction is irrelevant (as in [1] and many other places)—both components seem worth considering.

In a venerable tradition of non-classical mathematics, Brady has begun the bold dismantling of the classical edifice. He has guarded admirably against some difficult paradoxes by challenging various negation and implication rules. With others, he has shown how a weakened logic could tread where classicality cannot—with further yet to go. And so: could Hilbert's decision problem be solved in the affirmative after all? We *must* know; but *will* we? The universal logic project has come a long way. It remains open what is next.

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