On three-valued logics with the variable sharing property

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Abstract

The class of the logics free from paradoxes of relevance determined by all natural implicative expansions of Kleene's strong 3-valued matrix with two designated values is defined. These logics are free from paradoxes of relevance in the sense that they have the "variable sharing property". They have "natural conditionals" in the sense that the function defining them coincides with the classical function when restricted to the "classical values", satisfies *Modus Ponens* and, finally, assigns a designated value to a conditional whenever its antecedent and its consequence are assigned the same value. These logics are defined by using a "two-valued" overdetermined Belnap-Dunn semantics. Thus, the interpretation of the three values is crystalline. The logics here introduced enjoy the properties customarily demanded of many-valued implicative logics except, of course, the satisfaction of the rule "Verum ex quodlibet"

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1 Introduction

Since at least ten years ago, I have worked with José M. Méndez on topics defined by Ross T. Brady such as the "depth relevance condition" or the 4-valued logic BN4. It is an honor to present a paper in this special issue. Here, I define a Hilbert-style formulation for all 3-valued expansions of Kleene's strong logic with the variable sharing property. I use the method Ross T. Brady defined in [7].

Although I know that now he is not very fond of finite matrix semantics, I hope that he can see something valuable in my paper.

A propositional logic L has the variable sharing property (VSP) if in all L-theorems of implication form antecedent and consequent share at least a propositional variable. Given that in propositional logic the non-logical content is conveyed by propositional variables, if L is a propositional logic with the VSP, then it is free from "paradoxes of relevance" in the sense that L does not contain theorems of implication form where the semantical content of antecedent and consequent is disjoint. Anderson and Belnap consider the VSP a necessary property a relevant logic has to fulfill (cf. [1]), but some authors go so far as to consider that "the concept of relevant logic is coextensional with that of having the variable sharing property" (cf. [16, p. 28]).

The present paper investigates some properties of the elements of a certain class to be defined below, whose members are 3-valued logics with the VSP. We will use the term L*i*-logics (L*i*-logic) to refer to the logics in this class as a whole. But let us generally discuss the question of 3-valued logics and the VSP.

In algebraic and many-valued logic, the conditional (or "implication" as it was named in [36, pp. 227, ff.]) is traditionally required to meet the following strong condition c0: $a \to b = t$ iff $a \leq b$, where $\leq$ is the lattice order and t is the greatest element in the set of logical values, no other designated elements being considered in this set (cf., e.g., [36, p. 227] or [20, p. 179, ff.]). It results from the condition c0 that an *implicative logic* is defined as follows.

Definition 1.1 (Implicative logics)**.** A logic is implicative if it fulfills the ensuing conditions for any wffs A, B, C (cf. [20, pp. 179-180]; [35, p. 89]; [36, p, 228]):

c1. $A \to A$	Reflexivity
c2. $A \to B, A \Rightarrow B$	Modus Ponens
c3. $A \Rightarrow B \to A$	Veq
c4. $A \to B, B \to C \Rightarrow A \to C$	Transitivity
c5. $A \leftrightarrow B \Rightarrow \text{C}[A] \leftrightarrow \text{C}[A/B]$	Replacement

(Cf. Definition 2.1 on the logical language used in the paper; Veq abbreviates "Verum ex quodlibet" —"A true proposition follows from any proposition"—; $\text{C}[A]$ is a wff in which A appears; $\text{C}[A/B]$ is the result of substituting in $\text{C}[A]$ A by B in one or more places where A occurs; $A_1, ..., A_n \Rightarrow B$ means "if $A_1, ..., A_n$, then B".)

Wójcicki remarks "the set of conditions imposed on implicative logic can be treated as a set of minimal requirements that an adequate notion of implication has to satisfy" ([36, p. 227]). And he adds: "no logic that satisfies the Relevance

Principle is an implicative logic nor involves the implication connective characteristic of such logics" ([36, pp. 227-228]; by the Relevance Principle, Wójcicki refers to the VSP): Veq provides an infinity of wffs violating the VSP, the simplest of which is $q \to (p \to p)$, for distinct propositional variables p, q.

In this context, Tomova's notion of a "natural conditional" (cf. [33]) can be viewed as an attempt at extending the class of implicative logics worthy of the name beyond the restrictive limits imposed by condition c0. Now, let $\mathrm{MK3}_{\mathrm{II}}$ (resp., $\mathrm{MK3}_{\mathrm{I}}$) be Kleene's strong 3-valued matrix with two (resp., only one) designated values (cf. [14, §6.4]; Definition 2.2 below), a conditional is *natural* if the following three conditions are fulfilled (cf. Remark 2.5 below): (1) It coincides with the classical conditional when restricted to the "classical values" T and F; (2) it satisfies *Modus Ponens*; and (3) it is assigned a designated value whenever the value assigned to its antecedent is less than or equal to the value assigned to its consequent. It is clear that these conditions leave open the possibility of having more than one designated value, whence the class of implicative logics is considerably extended if the term " implicative" is understood according to Tomova's definition of a natural conditional instead of Definition 1.1, based on condition c0.

In [33], Tomova studies the lattice of the 30 natural implicative expansions of Kleene's strong 3-valued matrix (cf. Definition 2.2), 24 of these expanding $\mathrm{MK3}_{\mathrm{II}}$, while the remaining 6 are expansions of $\mathrm{MK3}_{\mathrm{I}}$. The logics determined by these 30 matrices are given Hilbert-style formulations in [26, 31], and it has to be remarked that, in addition to Łukasiewicz's Ł3, there are important logics among them such as the paraconsistent logic $\mathrm{G3}_{\mathrm{Ł}}^{\leq}$ (cf. [25]) and Pac (cf. [13]and references therein), the quasi-relevant logic RM3 (cf. [22] and references therein) or the intermediate logic with strong negation named " the logic of the three-element algebra" in [15]. Nevertheless, only four of these 30 logics are implicative logics in the sense of Definition 1.1: Ł3, the aforementioned logic $\mathrm{G3}_{\mathrm{Ł}}^{\leq}$ and the logics dubbed Lt3 and Lt6 in [31]. The remaining 26 logics either lack Veq or else *Replacement* (the rule $A \leftrightarrow B \Rightarrow \neg B \leftrightarrow \neg A$ fails).

But turning to the VSP, in [27, Appendix III], it is proved that none of the said 30 logics enjoys this property: $(A \wedge \neg A) \to (B \vee \neg B)$, the thesis form of the rule named "safety" in [10, p. 14], is a theorem of each one of them. Consequently, in the quoted paper [27], it is proposed to widen the class of natural implicative logics by modifying Tomova's definition: conditions c1 and c2 are maintained but condition c3 is replaced by c3′: a conditional is assigned a designated value whenever its antecedent and its consequent are assigned the same value (cf. Definition 2.4 below). It develops that, according to the modified definition of a natural conditional just given, there are exactly 108 natural implicative expansions of Kleene's strong 3-valued matrix, one half of them expands $\mathrm{MK3}_{\mathrm{I}}$, and the other half, $\mathrm{MK3}_{\mathrm{II}}$ (cf. [27]). Of these, in said paper [27], it is proved that just the 11 ones described in Definition 2.6 below determine logics with the VSP. However, after providing the tables defining them, these 11 logics are not discussed there any further. Next, in [30], it is shown that they are functionally equivalent. Concerning this question, let me then note just one point. In [27], it is proved that all natural implicative expansions of $\mathrm{MK3_I}$ are functionally equivalent to each other. Among these 54 expansions there are very well known logics such as Ł3 or $\mathrm{G3_Ł}$ (the positive fragment of G3 with Łukasiewicz-type negation) (cf., e.g., [13]). But also logics with lesser usual conditionals such as, for instance, the one defined in the ensuing table

$\to$	0	1	2
0	2	0	2
1	0	2	0
2	0	0	2

Surely, we are not willing to discard, say, Ł3 in favor of the logic just defined. In this sense, I do not think that the fact that the 11 aforementioned logics are functionally equivalent makes 10 of them dispensable in favor of the remaining one. Thus, the aim of this paper is to particularize each one of these logics. Especially, the aim of this paper is to give Hilbert-type formulations to the 11 L*i*-logics with the VSP and to highlight some interesting properties the L*i*-logics have, besides the VSP.

The Hilbert-style formulations presented below are defined by using a Belnap-Dunn "two-valued" semantics (BD-semantics), in particular, an overdetermined BD-semantics (o-semantics), characterized by the possibility of assigning truth, falsity or both truth and falsity to the formulas of a formal language. The idea is to simply translate the matrix semantics into an o-semantics and then prove completeness w.r.t. the o-semantics leaning upon a canonical model construction, whereas soundness is easily proved w.r.t. the matrix semantics. This strategy is based upon Brady's paper [7] as displayed in [26, 31]. Besides simplifying the soundness and completeness proofs, this strategy has the more important advantage of providing the logics it is applied to with a clear interpretation in terms of overdetermined models (we note that BD-semantics is not limited to the interpretation of logics with a Łukasiewicz-type negation: cf., e.g., [21]).

The introduction is ended with a remark. Then, we explain the structure of the paper. It is known that there are infinitely-many logics with the VSP (cf. [11]). Moreover, some many-valued logics with the VSP can be found in the literature. For instance, the logic determined by Belnap's eight-element matrix $\mathrm{M_0}$ (cf. [4]), axiomatized in [8]; for instance, the logic determined by Meyer's six-element Crystal lattice CL, also axiomatized in [8]. But it does not seem possible to interpret in an intuitive clear way the meaning of the logical values in these matrices. However, the meaning of the three values in the implicative expansions of MK3 introduced in the present paper is crystalline, since, as commented upon above, the matrix semantics is given equivalent BD-semantics, in particular, an o-semantics, whence the logical values can be interpreted as falsity (0), truth (2) and both truth and falsity (1). This fact together with the properties Li-logics enjoy, some of which have been remarked above, make of them, we think, interesting tools in contexts where relevance, paraconsistency and decidability are needed (notice that the more important relevant logics such as T, E and R are undecidable (cf. [34]).

The paper is organized as follows. In §2, all natural implicative expansions of MK3 determining logics with the VSP are defined. These logics (Li-logics) are formulated in the most general and unified way we have been capable of devising. But we remark that, as exemplified in the appendix, there are more simple and conspicuous ways of axiomatizing them if generality is dropped as a requisite. Also, some proof-theoretic properties of the Li-logics are highlighted in this second section of the paper. In §3, Belnap-Dunn semantics (BD-semantics) is provided for each one of the Li-logics and the soundness theorems is proved. In §4, completeness proofs are given by leaning on a canonical modal construction. As remarked above, we follow the strategy in [32] as applied in [7] and particularly displayed in [26] and [31]. We also note some remarkable properties of the Li-logics. In §5, we point out some concluding remarks on the results obtained and, finally, in the appendix, we prove some facts about the Li-logics claimed in some way or another throughout the paper.

2 Natural implicative expansions of MK3 determining logics with the variable sharing property

In this section, all natural 3-valued implicative expansions of Kleene's strong 3-valued matrix MK3 determining logics with the VSP are defined. We begin by defining MK3 and the notion of a natural conditional.

Definition 2.1 (Some preliminary notions)**.** The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ..$, and some or all of the following connectives $\rightarrow$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B etc. are metalinguistic variables. Then, logics are formulated as Hilbert-type axiomatic systems, the notions of "theorem" and "proof from a set of premises" being the usual ones, while the following notions are understood in a fairly standard sense (cf., e.g., [27] or [29]): "extension" and "expansion" of a given logic; "logical matrix" M and "M-interpretation", "M-consequence", "M-validity" and, finally, "M-determined" logic.

Kleene's strong matrix MK3 can be defined as follows (cf. [14, §64]).

Definition 2.2 (Kleene's strong 3-valued matrix)**.** The propositional language consists of the connectives $\wedge, \vee, \neg$. Kleene's strong 3-valued matrix, MK3 (our label), is the structure $(\mathcal{V}, D, \mathtt{F})$ where (1) $\mathcal{V} = \{0, 1, 2\}$ with $0 < 1 < 2$; (2) $D = \{1, 2\}$; (3) $\mathtt{F} = \{f_\wedge, f_\vee, f_\neg\}$ where $f_\wedge$ and $f_\vee$ are defined as the glb (or lattice meet) and the lub (or lattice joint), respectively, and $f_\neg$ is an involution with $f_\neg(2) = 0, f_\neg(0) = 2$ and $f_\neg(1) = 1$. We display the tables for $\wedge$, $\vee$ and $\neg$:

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\neg$	0
0	2
1	1
2	0

Remark 2.3 (On designated values in MK3)**.** Kleene does not seem to have considered designated values in [14, §64]. We use $2, 0$ and 1 instead of $\mathfrak{t}, \mathfrak{f}$ and $\mathfrak{u}$, respectively, used by Kleene. The former have been chosen in order to use the tester in [12], in case one is needed. Also, to put in connection the results in the present paper with previous work by us. (As remarked in the precedent section, by $\mathrm{MK3}_{\mathrm{I}}$ —resp., $\mathrm{MK3}_{\mathrm{II}}$—, we refer to MK3 with only one —resp., both— designated value(s).)

On the other hand, we set:

Definition 2.4 (Natural conditionals)**.** Let $\mathcal{V}$ and D be defined as in Definition 2.2. Then, an $f_\rightarrow$-function on $\mathcal{V}$ defines a natural conditional if the following conditions are satisfied:

1. $f_\rightarrow$ coincides with (the $f_\rightarrow$-function for) the classical conditional when restricted to the subset $\{0, 2\}$ of $\mathcal{V}$.
2. $f_\rightarrow$ satisfies Modus Ponens, that is, for any $a, b \in \mathcal{V}$, if $a \rightarrow b \in D$ and $a \in D$, then $b \in D$.
3. For any $a, b \in \mathcal{V}$, $a \rightarrow b \in D$ if $a = b$.

Remark 2.5 (Natural conditionals in Tomova's original paper)**.** We note that natural conditionals are defined in [33] exactly as in Definition 2.4 except for condition (3), which reads there as follows: for any $a, b \in \mathcal{V}$, $a \rightarrow b \in D$ if $a \leq b$.

Definition 2.6 (Natural 3-valued logics with the VSP)**.** In [27, Appendix III, Proposition C.9] (cf. Proposition 4.9), it is proved that the only natural implicative expansions of MK3 determining logics with the VSP are the ones built up with the conditional described by the following truth tables:

(t1)

$\to$	0	1	2
0	2	0	2
1	0	1	0
2	0	0	2

(t2)

$\to$	0	1	2
0	2	0	2
1	0	1	1
2	0	0	2

(t3)

$\to$	0	1	2
0	2	0	2
1	0	1	2
2	0	0	2

(t4)

$\to$	0	1	2
0	2	0	2
1	0	1	0
2	0	1	2

(t5)

$\to$	0	1	2
0	2	0	2
1	0	1	0
2	0	2	2

(t6)

$\to$	0	1	2
0	2	1	2
1	0	1	0
2	0	0	2

(t7)

$\to$	0	1	2
0	2	1	2
1	0	1	0
2	0	1	2

(t8)

$\to$	0	1	2
0	2	1	2
1	0	1	0
2	0	2	2

(t9)

$\to$	0	1	2
0	2	2	2
1	0	1	0
2	0	0	2

(t10)

$\to$	0	1	2
0	2	2	2
1	0	1	0
2	0	1	2

(t11)

$\to$	0	1	2
0	2	2	2
1	0	1	0
2	0	2	2

By Mi we refer to the implicative expansion of $\mathrm{MK3_{II}}$ (cf. Remark 2.3) built up by adding the $\to$-function described by table ti; by Li, we refer to the logic determined by Mi ($1 \leq i \leq 11$). Finally, we will use the term Li-logics (Li-logic) to generally refer to these eleven logics. Next, the Li-logics are defined, but firstly a common logic contained in each one of them, the logic b^3, is presented (the label b^3 is intended to abbreviate 'basic logic contained in all natural implicative expansions of MK3 with the VSP').

Definition 2.7 (The basic logic b^3)**.** The logic b^3 is axiomatized with the following axioms, rules of inference and metarule ($A_1, ..., A_n \Rightarrow B$ means "if $A_1, ..., A_n$, then B"):

Axioms:

a1. $A \to A$

a2. $(A \wedge B) \leftrightarrow (B \wedge A)$

a3. $[A \wedge (B \wedge C)] \leftrightarrow [(A \wedge B) \wedge C]$

a4. $[(A \vee B) \wedge (A \vee C)] \leftrightarrow [A \vee (B \wedge C)]$

a5. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$

a6. $A \leftrightarrow \neg\neg A$

a7. $A \vee \neg A$

Rules of inference:

$$\begin{aligned}
\text{Adjunction (Adj)}&: A, B \Rightarrow A \wedge B \\
\text{Modus Ponens (MP)}&: A \to B, A \Rightarrow B \\
\text{Elimination of conjunction (E}\wedge\text{)}&: A \wedge B \Rightarrow A, B \\
\text{Introduction of disjunction (I}\vee\text{)}&: A \Rightarrow A \vee B, B \vee A \\
\text{Conditioned introduction of conjunction (CI}\wedge\text{)}&: A \to B, A \to C \Rightarrow \\
&\quad A \to (B \wedge C) \\
\text{Elimination of disjunction (E}\vee\text{)}&: A \to C, B \to C \Rightarrow \\
&\quad A \vee B \to C \\
\text{Transitivity of } \leftrightarrow \text{ (Trans } \leftrightarrow\text{)}&: A \leftrightarrow B, B \leftrightarrow C \Rightarrow \\
&\quad A \leftrightarrow C \\
\text{Prefixing w.r.t. } \leftrightarrow \text{ (Pref}\leftrightarrow\text{)}&: A \leftrightarrow B \Rightarrow (C \to A) \leftrightarrow \\
&\quad (C \to B) \\
\text{Suffixing w.r.t. } \leftrightarrow \text{ (Suf}\leftrightarrow\text{)}&: A \leftrightarrow B \Rightarrow (A \to C) \leftrightarrow \\
&\quad (B \to C) \\
\text{Factor w.r.t. } \leftrightarrow \text{ (Fac}\leftrightarrow\text{)}&: A \leftrightarrow B \Rightarrow (C \wedge A) \leftrightarrow \\
&\quad (C \wedge B) \\
\text{Contraposition w.r.t. } \leftrightarrow \text{ (Con}\leftrightarrow\text{)}&: A \leftrightarrow B \Rightarrow \neg B \leftrightarrow \neg A
\end{aligned}$$

Metarule:

$$\text{Metarule: If } A, B \Rightarrow C \text{ then } D \vee A, D \vee B \Rightarrow D \vee C$$

In what follows, we prove some theorems and rules of b^3.

Proposition 2.8 (Some theorems and rules of b^3)**.** *The following are provable in b^3: (T1) $(A \leftrightarrow B) \leftrightarrow (B \leftrightarrow A)$; (T2) $A \leftrightarrow B \Rightarrow \neg A \leftrightarrow \neg B$; (T3) $(A \vee B) \leftrightarrow \neg(\neg A \wedge \neg B)$; (T4) $(A \vee B) \leftrightarrow (B \vee A)$; (T5) $A \leftrightarrow B \Rightarrow (A \wedge C) \leftrightarrow (B \wedge C)$; (T6) $A \leftrightarrow B \Rightarrow (C \vee A) \leftrightarrow (C \vee B)$; (T7) $A \leftrightarrow B \Rightarrow (A \vee C) \leftrightarrow (B \vee C)$ (T2 is Con$'\leftrightarrow$; T5 is Fac$'\leftrightarrow$; T6 is Summation w.r.t. $\leftrightarrow$ (Sum$\leftrightarrow$) and T7 is Sum$'\leftrightarrow$).*

Proof. T1: a2, Definition of $\leftrightarrow$. T2: T1, Con$\leftrightarrow$. T3: a5, a6, Con$\leftrightarrow$ and Trans$\leftrightarrow$. T4: a2, T3, Con$\leftrightarrow$ and Trans$\leftrightarrow$. T5: Fac$\leftrightarrow$ and a2. T6: Fac$\leftrightarrow$, Con$\leftrightarrow$, T3 and Trans$\leftrightarrow$. T7: T4 and T6. □

Proposition 2.9 (Replacement)**.** *Let Eb^3 be an extension of b^3. For any wffs A, B, $A \leftrightarrow B \Rightarrow C[A] \leftrightarrow C[A/B]$, where $C[A]$ is a wff in which A appears and $C[A/B]$ is the result of substituting A by B in $C[A]$ in one or more places where A occurs.*

Proof. Induction on the length of C[A] by using Trans↔, Fac↔, Fac′↔ (T5), Sum↔ (T6), Sum′↔ (T7), Pref↔, Suf↔, Con′↔ (T2) and Con↔. Let us prove one of the cases as a way of an example. Suppose C[A] is $D \to E$ and let C[A/B] be $D' \to E'$. Then, (by induction) we have (1) $D \leftrightarrow D'$ and (2) $E \leftrightarrow E'$. By 1 and Suf↔, we get (3) $(D' \to E) \leftrightarrow (D \to E)$; and by 2 and Pref↔, (4) $(D' \to E) \leftrightarrow (D' \to E')$. Finally, (5) $(D \to E) \leftrightarrow (D' \to E')$ is obtained by 3, 4 and Trans↔. □

Proposition 2.10 (More theorems of b^3)**.** *The following are provable in b^3: (T8) $(A \wedge B) \leftrightarrow \neg(\neg A \vee \neg B)$; (T9) $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$; (T10) $[A \vee (B \vee C)] \leftrightarrow [(A \vee B) \vee C]$; (T11) $[A \wedge (B \vee C)] \leftrightarrow [(A \wedge B) \vee (A \wedge C)]$.*

Proof. We use "Replacement" (Rep). T8: a5, a6. T9: T8, Con↔, a6. T10: a3, Con↔, T9, a6. T11: a4, Con↔, a5, T3, T9, a6. □

Next, the Li-logics are defined. As remarked above, it will be proved that the logic Li is determined by the matrix Mi ($1 \leq i \leq 11$).

Definition 2.11 (The Li-logics)**.** The Li-logics ($1 \leq i \leq 11$) are axiomatized by adding to b^3 the following axioms and/or rules:
L1:

A1. $(A \vee B) \vee (A \to B)$
A2. $(A \vee \neg B) \vee (A \to B)$
A3. $(\neg A \vee \neg B) \vee (A \to B)$
A4. $[(A \wedge \neg A) \wedge (B \wedge \neg B)] \to (A \to B)$
A5. $[\neg(A \to B) \wedge (\neg A \wedge \neg B)] \to (A \vee B)$
A6. $[\neg(A \to B) \wedge (\neg A \wedge B)] \to (A \vee \neg B)$
A7. $[\neg(A \to B) \wedge (A \wedge B)] \to (\neg A \vee \neg B)$
R1. $A \to B, \neg B \Rightarrow \neg A$
R2. $A \to B, A \wedge \neg A \Rightarrow \neg B$
R3. $A \to B, B \wedge \neg B \Rightarrow A$
R4. $B \wedge \neg B \Rightarrow \neg(A \to B)$
R5. $A \wedge \neg A \Rightarrow \neg(A \to B)$
R6. $A \wedge \neg B \Rightarrow \neg(A \to B)$

L2: A1, A5, A6, A7, R1, R3, R4, R5, R6 and

A8. $\neg B \vee (A \to B)$
A9. $[(A \wedge \neg A) \wedge B] \to (A \to B)$

L3: A1, A5, A8, A9, R1, R3, R4, R6 and

$$\text{A10. } [\neg(A \to B) \wedge B] \to \neg B$$

L4: A1, A2, A5, A6, A7, R2, R3, R4, R5, R6 and

$$\text{R7. } B \wedge \neg B, A \Rightarrow A \to B$$
$$\text{R8. } B \Rightarrow \neg A \vee (A \to B)$$

L5: A1, A2, A5, A6, R2, R3, R5, R7, R8 and

$$\text{R9. } B \wedge \neg B, \neg A \Rightarrow \neg(A \to B)$$
$$\text{R10. } A \wedge \neg B \Rightarrow B \vee \neg(A \to B)$$
$$\text{R11. } \neg(A \to B), B \Rightarrow \neg A$$

L6: A3, A5, A6, A7, R1, R2, R4, R5, R6 and

$$\text{A11. } A \vee (A \to B)$$
$$\text{R12. } B \wedge \neg B, \neg A \Rightarrow A \to B$$

L7: A5, A6, A7, A11, R2, R4, R5, R6, R8 and

$$\text{R13. } B \wedge \neg B \Rightarrow A \to B$$

L8: A5, A6, A11, R2, R5, R8, R9, R10, R11 and R13.
L9: A3, A7, A11, R1, R2, R5, R6, R12 and

$$\text{A12. } [\neg(A \to B) \wedge \neg A] \to A$$

L10: A7, A11, A12, R2, R5, R6, R8 and R13.
L11: A11, A12, R2, R5, R8, R10, R11 and R13.

3 Belnap-Dunn semantics for the Li-logics

Let T represent truth and F represent falsity. Belnap-Dunn semantics (BD-semantics) is characterized by the possibility of assigning T, F, both T and F or neither T nor F to the formulas of a given logical language (cf. [5, 6, 9, 10]). Concerning 3-valued logics, two variants of BD-semantics, overdetermined BD-semantics (o-semantics) and underdetermined BD-semantics (u-semantics) can be considered. Formulas can be assigned T, F or both T and F in the former; T, F or neither T nor F in the latter (cf. [26, 31]). U-semantics is especially adequate to 3-valued logics determined by matrices with only one designated value; o-semantics, for those determined by matrices where only one value is not designated.

Given an implicative expansion of MK3, M, with 1 and 2 as designated values, the idea for defining an o-semantics, M_o, equivalent to the matrix semantics based upon M is simple: a wff A is assigned both T and F in M_o iff A is assigned 1 in M. Next, A is assigned T (resp., F) in M_o iff it is not assigned 0 (resp., 2) in M. (Notice that, unlike in u-semantics, interpretation of formulas cannot be empty in o-semantics.)

The o-semantics equivalent to the matrix semantics based upon each one of the 11 matrices introduced above have been defined by translating said matrices into an o-semantics according to the simple intuitive ideas just exposed.

In the what follows, the notion of an Li-model and the accompanying notions of Li-consequence and Li-validity are defined. For each i ($1 \leq i \leq 11$), Li-models and said accompanying notions constitute an o-semantics (which will be referred to by Li-semantics) equivalent to the one based upon the matrix M_i in the sense explained above. It will be proved that the logic Li is sound and complete w.r.t. Li-models ($1 \leq i \leq 11$). We begin by defining the general notion of an Eb^3-model (models for extensions of the basic logic b^3).

Definition 3.1 (Eb^3-models)**.** An Eb^3-model is a structure (K, I) where (i) $K = \{\{T\}, \{F\}, \{T, F\}\}$, and (ii) I is an Eb^3-interpretation from the set of all wffs to K, this notion being defined according to the following conditions for each propositional variable p and wffs A, B: (1) $I(p) \in K$; (2a) $T \in I(\neg A)$ iff $F \in I(A)$; (2b) $F \in I(\neg A)$ iff $T \in I(A)$; (3a) $T \in I(A \wedge B)$ iff $T \in I(A)$ & $T \in I(B)$; (3b) $F \in I(A \wedge B)$ iff $F \in I(A)$ or $F \in I(B)$; (4a) $T \in I(A \vee B)$ iff $T \in I(A)$ or $T \in I(B)$; (4b) $F \in I(A \vee B)$ iff $F \in I(A)$ & $F \in I(B)$. There are five possibilities for assigning T to conditionals:

(5a1) $T \in I(A \to B)$ iff $[T \notin I(A)$ & $T \notin I(B)]$ or $[T \notin I(A)$ & $F \notin I(B)]$ or $[T \in I(A)$ & $F \in I(A)$ & $T \in I(B)$ & $F \in I(B)]$ or $[F \notin I(A)$ & $F \notin I(B)]$.

(5a2) $T \in I(A \to B)$ iff $F \notin I(B)$ or $[T \notin I(A)$ & $T \notin I(B)]$ or $[T \in I(A)$ & $F \in I(A)$ & $T \in I(B)]$.

(5a3) $T \in I(A \to B)$ iff $[T \notin I(A)$ & $T \notin I(B)]$ or $[T \notin I(A)$ & $F \notin I(B)]$ or $[T \in I(A)$ & $T \in I(B)$ & $F \in I(B)]$ or $[F \notin I(A)$ & $T \in I(B)]$.

(5a4) $T \in I(A \to B)$ iff $T \notin I(A)$ or $[F \in I(A)$ & $T \in I(B)$ & $F \in I(B)]$ or $[F \notin I(A)$ & $F \notin I(B)]$.

(5a5) $T \in I(A \to B)$ iff $T \notin I(A)$ or $[T \in I(B)$ & $F \in I(B)]$ or $[F \notin I(A)$ & $T \in I(B)]$.

There are also five possibilities for assigning F to conditionals:

(5b1) $F \in I(A \to B)$ iff $[T \in I(B)$ & $F \in I(B)]$ or $[T \in I(A)$ & $F \in I(A)]$ or $[T \in I(A)$ & $F \in I(B)]$.

(5b2) $F \in I(A \to B)$ iff $[T \in I(B)$ & $F \in I(B)]$ or $[T \in I(A)$ & $F \in I(B)]$.

(5b3) $F \in I(A \to B)$ iff $[F \in I(A)$ & $T \in I(B)$ & $F \in I(B)]$ or $[T \in I(A)$ & $F \in I(A)]$ or $[T \in I(A)$ & $T \notin I(B)]$.

(5b4) $F \in I(A \to B)$ iff $[T \in I(A)$ & $F \in I(A)]$ or $[T \in I(A)]$ & $F \in I(B)]$.

(5b5) $F \in I(A \to B)$ iff $[T \in I(A)$ & $F \in I(A)]$ or $[T \in I(A)]$ & $T \notin I(B)]$.

Then, Li-models ($1 \leq i \leq 11$) are defined as follows.

Definition 3.2 (Li-models)**.** For each i ($1 \leq i \leq 11$), an Li-model is an Eb^3-model with the following clauses for the conditional:

L1-models: (5a1) and (5b1).

L2-models: (5a2) and (5b1).

L3-models: (5a2) and (5b2).

L4-models: (5a3) and (5b1).

L5-models: (5a3) and (5b3).

L6-models: (5a4) and (5b1).

L7-models: (5a5) and (5b1).

L8-models: (5a5) and (5b3).

L9-models: (5a4) and (5b4).

L10-models: (5a5) and (5b4).

L11-models: (5a5) and (5b5).

Definition 3.3 (Li-consequence, Li-validity)**.** Let M be an Li-model ($1 \leq i \leq 11$). For any set of wffs Γ and wff A, $\Gamma \vDash_{\mathrm{M}} A$ (A is a consequence of Γ in the Li-model M) iff $T \in I(A)$ whenever $T \in I(\Gamma)$. Then, $\Gamma \vDash_{\mathrm{L}i} A$ (A is a consequence of Γ in Li-semantics) iff $\Gamma \vDash_{\mathrm{M}} A$ for each Li-model M $[T \in I(\Gamma)$ iff $\forall A \in \Gamma(T \in I(A)); F \in I(\Gamma)$ iff $\exists A \in \Gamma(F \in I(A))]$. In particular, $\vDash_{\mathrm{L}i} A$ (A is valid in Li-semantics) iff $\vDash_{\mathrm{M}} A$ for each Li-model M (i.e., iff $T \in I(A)$ for each Li-model M). (By $\vDash_{\mathrm{L}i}$ we shall refer to the relation just defined.)

Now, given the 11 matrices defined in Definition 2.6 together with Definition 3.2 and Definitions 2.1 and 3.3, we easily have:

Proposition 3.4 (Coextensiveness of $\vDash_{\mathrm{M}i}$ and $\vDash_{\mathrm{L}i}$)**.** *For any i $(1 \leq i \leq 11)$, set of wffs Γ and wff A, $\Gamma \vDash_{Mi} A$ iff $\Gamma \vDash_{Li} A$. In particular, $\vDash_{Mi} A$ iff $\vDash_{Li} A$.*

Proof. Cf., e.g., the proof of Proposition 7.4 in [26]. □

The proof of Proposition 3.4 is nothing but a mere formalization of the intuitive translation (commented upon above) of a semantics based upon the matrix Mi into the o-semantics formed by Li-models and the annexed notions of Li-consequence and Li-validity. Nevertheless, Proposition 3.4 greatly simplifies the soundness and completeness proofs, since we can focus on the relation $\vDash_{\mathrm{M}i}$ in the former case, while restricting our attention to the relation $\vDash_{\mathrm{L}i}$ in the latter one. Thus, let us prove soundness of the Li-logics w.r.t. the matrix semantics based upon the matrices Mi $(1 \leq i \leq 11)$. Then, soundness w.r.t. Li-semantics follows by Proposition 3.4.

Theorem 3.5 (Soundness of the Li-logics)**.** *For any i $(1 \leq i \leq 11)$, set of wffs Γ and wff A, if $\Gamma \vdash_{Li} A$ then (1) $\Gamma \vDash_{Mi} A$ and (2) $\Gamma \vDash_{Li} A$.*

Proof. (1) Given a particular logic Li and an Mi-interpretation I, it is easy to check the following facts: (a) Let R be a rule of Li. If I assigns a designated value to the premises of R, then its conclusion is also assigned a designated value. (b) Regarding the metarule, let $I(D \vee A) = I(D \vee B) = 1$ or 2 but $I(D \vee C) = 0$ for different wffs A, B, C, D. Then, it is clear that C is not an Mi-consequence of A, B (i.e., $A, B \Rightarrow C$ is falsified). (c) I assigns 1 or 2 to each one of the axioms of Li (in case a tester is needed the one in [12] can be used). (2) It is immediate by (1) and Proposition 3.4. □

4 Extension and primeness lemmas. Canonical models. Completeness

In this section the completeness of the Li-logics is proved by leaning upon a canonical model construction, We begin by defining the notion of an Eb^3-theory and the classes of Eb^3-theories of interest in the present paper (by EL, we refer to an extension or an expansion, as the case may be, of the logic L).

Definition 4.1 (Some preliminary notions)**.** Let L be an Eb^3-logic. An L-theory is a set of wffs closed under Adj, L-entailment (L-ent), all the rules and the metarule of b^3 and all the rules of L (A theory t is closed under L-ent iff whenever $A \rightarrow B$ is a theorem of L and $A \in t$, then $B \in t$). Then, an Eb^3-theory t is *regular* iff it contains all L-theorems; it is *prime* if $A \vee B \in t$, then $A \in t$ or $B \in t$, and, finally, it is *complete* if for every wff A, $A \in t$ or $\neg A \in t$.

Next, we sketch the framework of the completeness proofs.

A canonical Li-model is a structure (K, I_t) where K is defined as in Definition 3.1 and I_t is a t-interpretation built upon a prime, regular and complete Li-theory. A t-interpretation is a function from the set of all wffs to K defined as follows: for each wff A, $T \in I_t(A)$ iff $A \in t$, and $F \in I_t(A)$ iff $\neg A \in t$. Canonical Li-models are shown Li-models by proving that I_t fulfills the corresponding conditions in Definition 3.2. For instance, in the case of L4, we have to prove that the function I_t built upon the L4-theory t satisfies conditions (1), (2a), (2b), (3a), (3b), (4a), (4b), (5a3) and (5b1). Once canonical Li-models shown L-models, completeness is proved as follows. Let L be an Li-logic. Suppose that Γ is a set of wffs and A is a wff such that $\Gamma \nvdash_{\mathrm{L}} A$. Then, A does not belong to the set of consequences derivable in L from Γ (in symbols, $A \notin \mathrm{C}n\Gamma[\mathrm{L}]$). Now, the regular L-theory $\mathrm{C}n\Gamma[\mathrm{L}]$ is extended to a prime (hence, complete) theory t such that $A \notin t$. Then, t generates a t-interpretation I_t such that $T \in I_t(\Gamma)$ (since $T \in I_t\mathrm{C}n\Gamma[\mathrm{L}]$: $\mathrm{C}n\Gamma[\mathrm{L}] \subseteq t$) but $T \notin I_t(A)$, whence A does not follow from Γ in the canonical L-model, so $\Gamma \nvDash_{\mathrm{L}} A$.

Thus, in order to prove completeness, we need to prove: (a) every Li-theory without a given wff can be extended to a prime theory without said wff; (b) every canonical Li-model is in fact an Li-model.

Facts (a) and (b) are proved in what follows. Once these facts are proved, we think we are entitled to state the following theorem, where (2) follows from (1) by Proposition 3.4.

Theorem 4.2 (Completeness of the Li-logics)**.** *For any i $(1 \leq i \leq 11)$, set of wffs Γ and wff A, (1) if $\Gamma \vDash_{Li} A$, then $\Gamma \vdash_{Li} A$; (2) if $\Gamma \vDash_{Mi} A$, then $\Gamma \vdash_{Li} A$.*

So, let us proceed to the proof of facts (a) and (b).

As noted in the introduction to the paper, the completeness proofs we present follow the strategy set up in [32], as applied in [7], and particularly displayed in [26, 31]. According to this strategy, fact (a) can then be proved by using the notion of a "maximal set" as defined upon that of "disjunctive derivability" (disjunctive Eb^{3}derivability in the present case). In this sense, the proof can proceed similarly as, e.g., in [31, Section 6], since, as remarked there, the proof works for any logic including Anderson and Belnap's *First Degree Entailment Logic*, FDE. But each Li-logic contains FDE, as shown in the appendix (Proposition A.2). Concerning fact (b), the strategy exemplified in e.g., [26, 31] will also be used, but we need to be more specific about the details in this case. Firstly, Proposition 4.3 is proved. This proposition guarantees that the canonical interpretation of conjunction, disjunction and negation works in any Eb^3-logic.

Proposition 4.3 (Some properties of prime, regular Eb^3-theories)**.** *Let L be an Eb^3-logic and t be a prime, regular L-theory. Then, for any wffs A, B, (1) $A \wedge B \in t$*

iff $A \in t$ *and* $B \in t$*; (2)* $\neg(A \wedge B) \in t$ *iff* $\neg A \in t$ *or* $\neg B \in t$*; (3)* $A \vee B \in t$ *iff* $A \in t$ *or* $B \in t$*; (4)* $\neg(A \vee B) \in t$ *iff* $\neg A \in t$ *and* $\neg B \in t$*; (5)* $A \in t$ *iff* $\neg\neg A \in t$*; (6)* $A \in t$ *or* $\neg A \in t$*.*

Proof. (1) By E$\wedge$ and Adj. (2) By I$\vee$, T9 and primeness. (3) By I$\vee$ and primeness. (4) By E$\wedge$, Adj and a5. (5) By a6. (6) By a7 and primeness. (Notice that primeness is not needed in (1), (4) and (5), while regularity is only required in case (6).) □

In what follows, we proceed to the more laborious task of proving that the Li-canonical interpretation of the conditional also holds. In the first place, 5 basic extensions of b^3 are defined. These extensions are used to prove the fundamental properties of the conditional in prime, regular and complete theories built upon the Li-logics.

Definition 4.4 (Five basic extensions of b^3. I)**.** The logic b^3_i ($1 \leq i \leq 5$) is an extension of b^3 with the following axioms and rules (these axioms and rules are taken from Definition 2.11):

b^3_1:

A1. $(A \vee B) \vee (A \to B)$
A2. $(A \vee \neg B) \vee (A \to B)$
A3. $(\neg A \vee \neg B) \vee (A \to B)$
A4. $[(A \wedge \neg A) \wedge (B \wedge \neg B)] \to (A \to B)$
R1. $A \to B, \neg B \Rightarrow \neg A$
R2. $A \to B, A \wedge \neg A \Rightarrow \neg B$
R3. $A \to B, B \wedge \neg B \Rightarrow A$

b^3_2: A1, R1, R3 and

A8. $\neg B \vee (A \to B)$
A9. $[(A \wedge \neg A) \wedge B] \to (A \to B)$

b^3_3: A1, A2, R2, R3 and

R7. $B \wedge \neg B, A \Rightarrow A \to B$
R8. $B \Rightarrow \neg A \vee (A \to B)$

b^3_4: A3, R1, R2 and

A11. $A \vee (A \to B)$
R12. $B \wedge \neg B, \neg A \Rightarrow A \to B$

b^3_5: A11, R2, R8 and

R13. $B \wedge \neg B \Rightarrow A \to B$

We prove:

Proposition 4.5 (The conditional in prime regular Eb_i^3-theories)**.** *Let L be an Eb_i^3-theory where b_i^3 $(1 \le i \le 5)$ will refer in each case to one of the extensions of b^3 displayed in Definition 4.4. And let t be a prime, regular L-theory. We have the following properties Pi $(1 \le i \le 5)$:*

P1 (Eb_1^3-logics): $A \to B \in t$ *iff* $[A \notin t \,\&\, B \notin t]$ *or* $[A \notin t \,\&\, \neg B \notin t]$ *or* $[A \in t \,\&\, \neg A \in t \,\&\, B \in t \,\&\, \neg B \in t]$ *or* $[\neg A \notin t \,\&\, \neg B \notin t]$.

P2 (Eb_2^3-logics): $A \to B \in t$ *iff* $\neg B \notin t$ *or* $[A \notin t \,\&\, B \notin t]$ *or* $[A \in t \,\&\, \neg A \in t \,\&\, B \in t]$.

P3 (Eb_3^3-logics): $A \to B \in t$ *iff* $[A \notin t \,\&\, B \notin t]$ *or* $[A \notin t \,\&\, \neg B \notin t]$ *or* $[A \in t \,\&\, B \in t \,\&\, \neg B \in t]$ *or* $[\neg A \notin t \,\&\, B \in t]$.

P4 (Eb_4^3-logics): $A \to B \in t$ *iff* $A \notin t$ *or* $[\neg A \in t \,\&\, B \in t \,\&\, \neg B \in t]$ *or* $[\neg A \notin t \,\&\, \neg B \notin t]$.

P5 (Eb_5^3-logics): $A \to B \in t$ *iff* $A \notin t$ *or* $[B \in t \,\&\, \neg B \in t]$ *or* $[\neg A \notin t \,\&\, B \in t]$.

Proof. Eb_1^3-logics. (a) Suppose (1) $A \to B \in t$, and, for reductio, (2) $[A \in t$ or $B \in t]$ & $[A \in t$ or $\neg B \in t]$ & $[A \notin t$ or $\neg A \notin t$ or $B \notin t$ or $\neg B \notin t]$ & $[\neg A \in t$ or $\neg B \in t]$. There are 32 subcases to consider. The first 16 are the following:

1. $A \in t \,\&\, A \in t \,\&\, A \notin t \,\&\, \neg A \in t$.
2. $A \in t \,\&\, A \in t \,\&\, \neg A \notin t \,\&\, \neg A \in t$.
3. $A \in t \,\&\, A \in t \,\&\, B \notin t \,\&\, \neg A \in t$.
4. $A \in t \,\&\, A \in t \,\&\, \neg B \notin t \,\&\, \neg A \in t$.
5. $A \in t \,\&\, A \in t \,\&\, A \notin t \,\&\, \neg B \in t$.
6. $A \in t \,\&\, A \in t \,\&\, \neg A \notin t \,\&\, \neg B \in t$.
7. $A \in t \,\&\, A \in t \,\&\, B \notin t \,\&\, \neg B \in t$.
8. $A \in t \,\&\, A \in t \,\&\, \neg B \notin t \,\&\, \neg B \in t$.
9. $A \in t \,\&\, \neg B \in t \,\&\, A \notin t \,\&\, \neg A \in t$.
10. $A \in t \,\&\, \neg B \in t \,\&\, \neg A \notin t \,\&\, \neg A \in t$.
11. $A \in t \,\&\, \neg B \in t \,\&\, B \notin t \,\&\, \neg A \in t$.

12. $A \in t \,\&\, \neg B \in t \,\&\, \neg B \notin t \,\&\, \neg A \in t$.
13. $A \in t \,\&\, \neg B \in t \,\&\, A \notin t \,\&\, \neg B \in t$.
14. $A \in t \,\&\, \neg B \in t \,\&\, \neg A \notin t \,\&\, \neg B \in t$.
15. $A \in t \,\&\, \neg B \in t \,\&\, B \notin t \,\&\, \neg B \in t$.
16. $A \in t \,\&\, \neg B \in t \,\&\, \neg B \notin t \,\&\, \neg B \in t$.

Now, subcases 1, 2, 5, 8, 9, 10, 12, 13 and 16 are impossible, since each one of them contains a contradiction; subcases 3, 7, 11 and 15 are also impossible, since they contravene MP. So, we are left with subcases 4, 6 and 14, which are proved as follows: subcase 4, by R2; and subcases 6 and 14, by R1. Let us prove, for example, subcase 4. By the hypothesis (1) and Adj, we have $(A \to B), (A \wedge \neg A) \in t$, whence, by R2, $\neg B \in t$ follows, contradicting 4.

Concerning the remaining 16 subcases, they are exactly as 1-16 above, except that $A \in t$ (the first member in each conjunction) is replaced by $B \in t$. Let us name $1'$-$18'$ these remaining 18 subcases. Then, $1'$, $2'$, $3'$, $5'$, $7'$, $8'$, $10'$, $11'$, $12'$, $15'$ and $16'$ contain a contradiction. Next, subcases $6'$ and $14'$ are proved by R1; subcase $4'$, by R2 and finally, subcases $9'$ and $13'$, by R3.

(b) Suppose (1) $A \notin t \;\&\; B \notin t$ or (2) $A \notin t \,\&\, \neg B \notin t$ or (3) $A \in t \,\&\, \neg A \in t \,\&\, B \in t \,\&\, \neg B \in t$ or (4) $\neg A \notin t \,\&\, \neg B \notin t$. Then $A \to B \in t$ follows by A1, A2, A4 and A3, respectively. Consider, for example, (2). By A2 and regularity of t, we have $(A \vee \neg B) \vee (A \to B) \in t$, whence, by primeness of t, we get $A \to B \in t$.

The proof for the rest of the Eb_i^3-logics is similar. It is remarkable that the L-theory t needs not be complete or consistent in any sense of the term. (Anyway, notice that prime and regular Eb_i^3-theories are complete by a7.) □

In what follows, 5 more basic extensions of b^3 are defined. Similarly as it has been the case with b1-b5 and the fundamental properties of the conditional just proved, the new extensions are used to prove essential properties of negated conditionals in prime, regular and (now also) complete theories built upon the L*i*-logics.

Definition 4.6 (Five basic extensions of b^3. II)**.** The logic b_i^3 $(6 \leq i \leq 10)$ is an extension of b^3 with the following axioms and rules (these axioms and rules are taken from Definition 2.11):

b^3_6:

A5. $[\neg(A \to B) \wedge (\neg A \wedge \neg B)] \to (A \vee B)$
A6. $[\neg(A \to B) \wedge (\neg A \wedge B)] \to (A \vee \neg B)$
A7. $[\neg(A \to B) \wedge (A \wedge B)] \to (\neg A \vee \neg B)$
R4. $B \wedge \neg B \Rightarrow \neg(A \to B)$
R5. $A \wedge \neg A \Rightarrow \neg(A \to B)$
R6. $A \wedge \neg B \Rightarrow \neg(A \to B)$

b^3_7: A5, R4, R6 and

A10. $[\neg(A \to B) \wedge B] \to \neg B$

b^3_8: A5, A6, R5 and

R9. $B \wedge \neg B, \neg A \Rightarrow \neg(A \to B)$
R10. $A \wedge \neg B \Rightarrow B \vee \neg(A \to B)$
R11. $\neg(A \to B), B \Rightarrow \neg A$

b^3_9: A7, R5, R6 and

A12. $[\neg(A \to B) \wedge \neg A] \to A$

b^3_{10}: A12, R5, R10 and R11.

We prove:

Proposition 4.7 (The conditional in prime regular and complete Eb^3_i-theories)**.** *Let L be an Eb^3_i-logic where b^3_i ($6 \leq i \leq 10$) will refer in each case to one of the extensions of b^3 displayed in Definition 4.6. And let t be a prime, regular and complete L-theory. We have the following properties Pi ($6 \leq i \leq 10$):*

P6 (Eb^3_6-logics): $\neg(A \to B) \in t$ *iff* $[B \in t \;\&\; \neg B \in t]$ *or* $[A \in t \;\&\; \neg A \in t]$ *or* $[A \in t \;\&\; \neg B \in t]$.

P7 (Eb^3_7-logics): $\neg(A \to B) \in t$ *iff* $[B \in t \;\&\; \neg B \in t]$ *or* $[A \in t \;\&\; \neg B \in t]$.

P8 (Eb^3_8-logics): $\neg(A \to B) \in t$ *iff* $[\neg A \in t \;\&\; B \in t \;\&\; \neg B \in t]$ *or* $[A \in t \;\&\; \neg A \in t]$ *or* $[A \in t \;\&\; B \notin t]$.

P9 (Eb^3_9-logics): $\neg(A \to B) \in t$ *iff* $[A \in t \;\&\; \neg A \in t]$ *or* $[A \in t \;\&\; \neg B \in t]$.

P10 (Eb^3_{10}-logics): $\neg(A \to B) \in t$ *iff* $[A \in t \;\&\; \neg A \in t]$ *or* $[A \in t \;\&\; B \notin t]$.

Proof. Eb_6^3-logics. (a) Suppose (1) $\neg(A \to B) \in t$, and, for reductio, (2) $[B \notin t$ or $\neg B \notin t]$ & $[A \notin t$ or $\neg A \notin t]$ & $[A \notin t$ or $\neg B \notin t]$. There are 8 subcases to consider:

1. $B \notin t$ & $A \notin t$ & $A \notin t$.
2. $B \notin t$ & $A \notin t$ & $\neg B \notin t$.
3. $B \notin t$ & $\neg A \notin t$ & $A \notin t$.
4. $B \notin t$ & $\neg A \notin t$ & $\neg B \notin t$.
5. $\neg B \notin t$ & $A \notin t$ & $A \notin t$.
6. $\neg B \notin t$ & $A \notin t$ & $\neg A \notin t$.
7. $\neg B \notin t$ & $\neg A \notin t$ & $A \notin t$.
8. $\neg B \notin t$ & $\neg A \notin t$ & $\neg B \notin t$.

Now, subcases 2, 3, 4, 6 and 7 are impossible, since they contradict the fact that t is complete: so, we are left with subcases 1, 5 and 8, which are proved by A5, A6 and A7, respectively. Let us prove, for instance, subcase 1. By completeness of t, we have $\neg A \in t$ and $\neg B \in t$, which together with the hypothesis (1) ($\neg(A \to B) \in t$) and A5 give us $A \vee B \in t$, whence by primeness, $A \in t$ or $B \in t$ follows, contradicting 1.

(b) Suppose (1) $B \in t$ & $\neg B \in t$ or (2) $A \in t$ & $\neg A \in t$ or (3) $A \in t$ & $\neg B \in t$. Then $\neg(A \to B) \in t$ follows by R4, R5 and R6, respectively.

The proof for the rest of the Eb_i^3-logics ($7 \leq i \leq 10$) is similar. □

Now, given Propositions 4.5 and 4.7, we have the following fact concerning the Li-logics.

Proposition 4.8 (The conditional in prime, regular, complete Li-theories)**.** *By Li ($1 \leq i \leq 11$), we refer to the extensions of b^3 in Definition 2.11; by Pi ($1 \leq i \leq 10$), to the properties in Propositions 4.5 and 4.7. Now, let t be a prime, regular and complete Li-theory. We have:*

1. *t is an L1-theory: t has P1 and P6.*
2. *t is an L2-theory: t has P2 and P6.*
3. *t is an L3-theory: t has P2 and P7.*
4. *t is an L4-theory: t has P3 and P6.*

5. *t is an L5-theory: t has P3 and P8.*

6. *t is an L6-theory: t has P4 and P6.*

7. *t is an L7-theory: t has P5 and P6.*

8. *t is an L8-theory: t has P5 and P8.*

9. *t is an L9-theory: t has P4 and P9.*

10. *t is an L10-theory: t has P5 and P9.*

11. *t is an L11-theory: t has P5 and P10.*

Proof. Immediate from Propositions 4.5 and 4.7. Take, for instance, case 7: it follows from the fact that L7 is an Eb^3_5-logic and an Eb^3_6-logic. The rest of the cases is proved similarly. □

From Proposition 4.8, it follows immediately that the canonical interpretation of the Li-conditionals holds. For instance, consider L2. The L2-conditional is interpreted (cf. Definition 3.1) according to clauses 5a2, i.e., $T \in I(A \to B)$ iff $F \notin I(B)$ or $[T \notin I(A)\ \&\ T \notin I(B)]$ or $[T \in I(A)\ \&\ F \in I(A)\ \&\ T \in I(B)]$ and 5b1, i.e., $F \in I(A \to B)$ iff $[T \in I(B)\ \&\ F \in I(B)]$ or $[T \in I(A)\ \&\ F \in I(A)]$ or $[T \in I(A)\ \&\ F \in I(B)]$. Well then, if t is a prime, regular and complete L2-theory, it follows from Proposition 4.8 that t has P2, i.e., $A \to B \in t$ iff $\neg B \notin t$ or $[A \notin t\ \&\ B \notin t]$ or $[A \in t\ \&\ \neg A \in t\ \&\ B \in t]$ and P6, i.e., $\neg(A \to B) \in t$ iff $[B \in t\ \&\ \neg B \in t]$ or $[A \in t\ \&\ \neg A \in t]$ or $[A \in t\ \&\ \neg B \in t]$.

The section is ended by noting some properties of the Li-logics:

1. *Weak implicative logics*: we know that *replacement* is a property of each Li-logic (cf. Proposition 2.9). Now, let $k \in \{1, 2, 3, 5, 6, 8, 9, 10, 11\}$. L$k$ fulfills all properties required of implicative logics in, e.g., [20] or [36], except, of course, that of having the rule Veq (cf. Definition 1.1; L4 and L7 are excluded, since they lack the rule transitivity). Could we then say that they are *weak implicative logics*, that is, logics with conditions (c1), (c2), (c4) and (c5) but without c(3)? (Cf. Definition 1.1.)

2. *Strong and expressive logics*: along section 2, it has been shown that, from a syntactical point of view, the Li-logics are strong logics. In addition, they have considerable expressive power (cf. [30]) For example, classical possitive logic is definable in each one of them (cf. [30, Proposition 3.13]).

3. *Variable sharing property*:

Proposition 4.9 (The Li-logics have the VSP)**.** *Let L be an* Li*-logic. Then L has the VSP.*

Proof. (a) Let L be any Li-logic except L2 or L3 and let ML be the implicative expansion of MK3 determining L. Suppose that there are wffs A, B such that $A \to B$ is an L-theorem but A and B do not share propositional variables. Let I be an ML-interpretation assigning 1 (resp., 0) to each propositional variable in A (resp., B). Then $I(A) = 1$ and $I(B) \in \{0, 2\}$ since $\{1\}$ and $\{0, 2\}$ are closed under $\to, \wedge, \vee$ and $\neg$. Consequently, $I(A \to B) = 0$ whence $A \to B$ is not an L-theorem by soundness (cf. Theorem 3.5). (b) Let L be L2 or L3. The proof is similar to that of case (a) by using now the fact that $f_{\to}(0, 1) = f_{\to}(2, 1) = 0$. □

4. *Paraconsistency*:

   Proposition 4.10 (The Li-logics are paraconsistent)**.** *Let L be an* Li*-logic. Then, the rule ECQ (i.e., $A \wedge \neg A \Rightarrow B$) fails in L.*

   Proof. Let ML be the implicative expansion of MK3 determining L, and I be an ML-interpretation such that for distinct propositional variables p, q, $I(p) = 1$ and $I(q) = 0$. Then, $I(p \wedge \neg p) = 1$ but $I(q) = 0$. □

5. *Comparison with other relevant logics*: it is remarkable that, although all Li-logics contain Anderson and Belnap's *First Degree Entailment Logic* FDE, none of them contains Routley and Meyer's basic logic B (cf. Proposition A.2 in the appendix). Nevertheless, no Li-logic is contained in strong logics such as RM3 (the 3-valued extension of the semi-relevant logic R-Mingle) or the 6-valued relevant logic CL (the logic determined by Meyer's *Crystal lattice* CL —cf. Proposition A.4, in the appendix.) Therefore, none of the Li-logics is contained in Anderson and Belnap's logic of relevant implication R.

6. *On the H-formulations of the* Li*-logics*: the Li-logics have been axiomatized in a general and unified way, but they can be given simpler axiomatizations. For instance, L1 is alternatively axiomatized in Proposition A.5 in the appendix. (Moreover, we note that some of the rules R1-R13 can be strengthened to the corresponding thesis form; also, that some of the Li-logics contain relatively strong rules, for instance, the rules Suffixing and Prefixing —cf. A1 in the appendix— are valid in L1 and L7. These facts can be used to provide simpler formulations of the Li-logics.)

7. *On the axioms and rules of the Li-logics*: let us now take a look at the axioms and rules of the Li-logics. Those of b^3 are reasonable enough: the axioms and rules of b^3 are clearly provable in the basic logic B. Moreover, the metarule is a rule of "disjunctive B", B^d (cf., e.g., [8]). But concerning A1-A12 and R1-R13, the situation is entirely different. For example, only R1 (the rule *Modus Tollens*), R6 and R10, from the list A1-A12, R1-R13, are provable in the logic of relevant implication R. And nevertheless the items in said list are not freak axioms and rule schemes. Actually, they are weakenings of well known, if debatable, axioms and rules. In particular, we have: A4, A9, R2, R3, R4, R5, R7, R9, R12 and R13 are related to the "Ex contradictione quodlibet" (ECQ) axiom, $(A \wedge \neg A) \to B$; A1 and A2 are related to the disjunctive Peirce's law A11, $A \vee (A \to B)$; A3, A8 and R8 to the "Verum ex quodlibet" (VEQ) axiom, $A \to (B \to A)$; A5, A6, A7, A10, A12 and R11 to $\neg(A \to B) \to (A \wedge \neg B)$; R6 and R10 to $(A \wedge \neg B) \to \neg(A \to B)$, and finally, R1 is the rule "Modus Tollens". Anyway, some of the schemes characteristic of the Li-logics are named "minglish axioms" in [32] because of their relationship with the "mingle axiom", $A \to (A \to A)$, and in fact, a few are provable in CL and RM3. For instance, A1 through A12, R1, R4, R6, R9, R10 and R12 are provable in RM3.

5 Concluding remarks

1. In this paper, we have axiomatized all logics with the VSP determined by natural implicative expansions of Kleene's strong 3-valued matrix either with only one ($MK3_I$) or with two designated values ($MK3_{II}$). We hope that these logics be useful in contexts when relevance paraconsistency and decidability are required. There are other 3-valued logics with the VSP determined by implicative expansions of $MK3_{II}$, but these fail to be natural in the generous sense of Definition 2.4 (cf. [29, Proposition 7.12]).

2. Moreover, consider two interesting alternatives to Łukasiewicz-type negation, i.e., Gödel-type negation (G-negation), and dual Gödel-type negation (dG-negation), which in 3-valued logic can be given as follows (cf. [28, 23] and the references in these items):

G-negation

	$\neg$
0	2
1	0
2	0

dG-negation

	$\neg$
0	2
1	2
2	0

Let $MK3_G$ (resp., $MK3_{dG}$) be the result of replacing Łukasiewicz-type negation in $MK3_I$ (resp., $MK3_{II}$) by G-negation (resp., dG-negation). Then, there are no implicative expansions with the VSP of neither $MK3_G$ nor $MK3_{dG}$: for distinct propositional variables p, q, use the wff $\neg(p \to p) \to \neg(q \to q)$ (resp., $\neg(p \vee \neg p) \to \neg(q \wedge \neg q)$) in case of $MK3_G$ (resp., $MK3_{dG}$).

3. The 3-valued logics with the VSP we have defined (the Li-logics) fulfill all conditions required of implicative logics in, e.g., [20] or [36], except, of course, that of having the rule VEQ (cf. the introduction to the paper).

4. It is known that there are infinitely many logics with the VSP (cf. [11]). Moreover there are, for instance, strong 6-valued or 8-valued logics with the VSP (cf. the introduction to the paper). However, the truth-values these logics are interpreted with are not easy to elucidate intuitively. In contrast, the 3 truth-values used to interpret the Li-logics are transparent, most of all as rendered by BD-semantics. Additionally, the Li-conditional (or Li-implication) is clearly interpretable from an intuitive point of view: it translates that of a weak implicative logic with the supplementary property of having the VSP (cf. items 1 and 3 at the end of the preceding section).

5. As shown above, the Li-logics have interesting properties besides the VSP such as paraconsistency or Wójcicki and Rasiowa conditions (except Veq).

6. Also, in the appendix, it is shown that the Li-logics maintain peculiar relations to standard relevance logics: none of them includes such a weak logic as Routley and Meyer's basic logic B but, on the other hand, the sometimes referred to as the strongest logic in the relevance logic family, i.e., RM3 (cf. [22] and references therein) does not include any of the Li-logics.

7. As explained in §2, we have tried to axiomatize the Li-logics in a general and unified way. Nevertheless, these logics admit more simple and conspicuous axiomatizations. In the appendix, an example is provided in the case of L1. The rest of the Li-logics can be more simply axiomatized, in a similar way.

8. Finally, we note that the Li-logics could perhaps be given cut-free Gentzen-systems (resp., natural deduction formulations) following the methods in [2, 3] (resp., [18]). Concerning the comparison between these methods, cf. [19, §8] and [24, §6].

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A Appendix. Some additional facts about the Li-logics

Consider the following relevant logics.

Definition A.1 (The logics FDE and B)**.** Anderson and Belnap's First Degree Entailment logic FDE can be given a Hilbert-style system with the following rules (cf. [17]and references therein):

$$\frac{A \wedge B}{B} \quad (R1) \qquad \frac{(A \vee B) \wedge (A \vee C)}{A \vee (B \wedge C)} \quad (R9)$$

$$\frac{A \wedge B}{A} \quad (R2) \qquad \frac{A \vee C}{\neg\neg A \vee C} \quad (R10)$$

$$\frac{A, B}{A \wedge B} \quad (R3) \qquad \frac{\neg\neg A \vee C}{A \vee C} \quad (R11)$$

$$\frac{A}{A \vee B} \quad (R4) \qquad \frac{\neg(A \vee B) \vee C}{(\neg A \wedge \neg B) \vee C} \quad (R12)$$

$$\frac{A \vee B}{B \vee A} \quad (R5) \qquad \frac{(\neg A \wedge \neg B) \vee C}{\neg(A \vee B) \vee C} \quad (R13)$$

$$\frac{A \vee A}{A} \quad (R6) \qquad \frac{\neg(A \wedge B) \vee C}{(\neg A \vee \neg B) \vee C} \quad (R14)$$

$$\frac{A \vee (B \vee C)}{(A \vee B) \vee C} \quad (R7) \qquad \frac{(\neg A \vee \neg B) \vee C}{\neg(A \wedge B) \vee C} \quad (R15)$$

$$\frac{A \vee (B \wedge C)}{(A \vee B) \wedge (A \vee C)} \quad (R8)$$

On the other hand, Routley and Meyer's basic logic B can be axiomatized as follows (cf. [32, Chapter 4]). Axioms: (α1) $(A \wedge B) \to A$; $(A \wedge B) \to B$; (α2) $A \to (A \vee B)$; $B \to (A \vee B)$; (α3) $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$; ($\alpha$4) $A \leftrightarrow \neg\neg A$; ($\alpha$5) $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$; ($\alpha$6) $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$. Rules: Adj, MP, Suffixing (Suf): $A \to B \Rightarrow (B \to C) \to (A \to C)$; Prefixing (Pref) $B \to C \Rightarrow (A \to B) \to (A \to C)$; Contraposition (Con): $A \to B \Rightarrow \neg B \to \neg A$ (it is easy to prove that FDE is contained in B).

We have:

Proposition A.2 (The Li-logics, FDE and B)**.** *(a) FDE is contained in each Li-logic. (b) B is contained in none of the Li-logics.*

Proof. (a) It immediately follows by inspection of the formulations of FDE and b^3 (cf. also Propositions 2.8, 2.9 and 2.10). (b) It is easy to check that the rule Con fails in all Mi-matrices except M1 but M1 falsifies $\alpha 1$ and $\alpha 2$ (in case a tester is needed, that in [12] can be used). □

FDE and B are weak logics. Next, some strong relevant logics are defined.

Definition A.3 (R, RM3 and CL)**.** Anderson and Belnap's logic of relevant implication, R, can be axiomatized as follows (cf. [1], [32, Chapter 4]). Axioms: $\alpha 1$-$\alpha 6$ of B, and ($\alpha 7$) $[A \to (A \to B)] \to (A \to B)$; ($\alpha 8$) $A \to [(A \to B) \to B]$; ($\alpha 9$) $(A \to B) \to [(B \to C) \to (A \to C)]$ and ($\alpha 10$) $(A \to B) \to (\neg B \to \neg A)$. Rules: Adj and MP. The 3-valued extension of R-Mingle, RM3, can be axiomatized by adding to R the axioms ($\alpha 11$) $A \to (A \to A)$ and ($\alpha 12$) $A \vee (A \to B)$ (cf. [1]). Finally, Meyer's *Crystal lattice* 6-valued logic CL is axiomatized when adding to R $\alpha 12$ and ($\alpha 13$) $(\neg A \wedge B) \to [(\neg A \to A) \vee (A \to B)]$ (cf. [8]). We note that RM3 is the logic determined by the matrix MRM3, the implicative expansion of MK3_{II} resulting from adding the conditional described by the following truth-table:

$\to$	0	1	2
0	2	2	2
1	0	1	2
2	0	0	2

(We note that CL is included in RM3: it is immediately checked that $\alpha 13$ is verified by MRM3.)

We have:

Proposition A.4 (The Li-logics, R, RM3 and CL)**.** *None of the Li-logics is contained in RM3. Therefore, none of the Li-logics is contained in R or in CL.*

Proof. (Cf. §2.) It is immediate to check that R5, $A \wedge \neg A \Rightarrow \neg(A \to B)$, holds in all L$i$-logics except L3 where, nevertheless, R3, $A \to B, B \wedge \neg B \Rightarrow A$ is provable. But R5 and R6 fail in RM3. □

The appendix is ended with a proposition on the axiomatization of L1.

Proposition A.5 (Alternative axiomatization of L1)**.** *L1 can be formulated as follows. Axioms: a1-a7, A1, A2, A4, A5, A6 and the contraposition axiom A6′: $(A \to B) \to (\neg B \to \neg A)$. Rules: Adj, MP, E∧, I∨, CI∧, E∨, Fac↔, Suf→ (i.e., $A \to B \Rightarrow (B \to C) \to (A \to C)$), and Pref→ (i.e., $B \to C \Rightarrow (A \to B) \to (A \to C)$); also, R2, R4 and R6.*

Proof. Firstly, notice that A6′, Suf→ and Pref→ are verified by ML1 (however, Fac→, $A \to B \Rightarrow (C \wedge A) \to (C \wedge B)$, is falsified). Next, we have to prove that Trans↔, Suf↔, Pref↔, Con↔, R1, R3, R5, A3 and A7 are provable in the new formulation of L1. Well then, Trans→ (i.e., $A \to B, B \to C \Rightarrow A \to C$) is immediate by Suf→ (or Pref→), and Trans↔, Suf↔, Pref↔ and Con↔ are, in their turn, immediate by Trans→, Suf→, Pref→ and A6′. Concerning the specific axioms and rules of L1 we have: R1: immediate by A6′; R5: by R4, A6′ and a6. R3: by R2, A6′ and a6. A3: by A1 and A6′. A7: by A5, A6′ and a6. □

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