A Normalized Natural Deduction System for the Logic MC

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Abstract

A normalized natural deduction system for the weak relevant logic DW was created in Brady [1996a] from which the decidability of the logic was determined. Here, we make adjustments to this natural deduction system and its decidability argument to fit the logic MC of meaning containment. The two key adjustments to be made are the removal of the distribution axiom and the introduction of conjunctive syllogism. The decidability result is new in that the standard methods of using a cut-free Gentzen system or a filtration of Routley-Meyer semantics do not apply.

The logic MC of meaning containment was set up as such in Brady and Meinander [2013] justifiably removing the distribution axiom, $A\&(B \vee C) \to (A\&B) \vee (A\&C)$, from the earlier logic $\mathrm{DJ^d}$ of Brady [1996]. ($\mathrm{DJ^d}$ also features in the book *Universal Logic* (Brady [2006]), where it was also motivated as a logic based on meaning containment.) It was argued in [2013] that the distribution axiom was to be removed by showing that it did not follow from the meanings of conjunction and disjunction and thus should not be included in a logic based on meaning containment. The quantificational extension MCQ was also given in [2013] by a corresponding addition of axioms and rules, which does not include the quantificational distribution axioms. However, in this paper we just treat the logic MC, axiomatized below in §1, leaving the corresponding treatment of MCQ for a subsequent paper, due to the length and intensity of this one.

We will start by determining the normalized natural deduction system for the logic MC and then go on to prove its key property of decidability. To do this, we will prove the subformula property which states that only subformulae of the formula under test for decidability can occur in its proof. We then prove a corollary of this which identifies the depths of subformulae of the formula under test with the depths of subproofs in which such subformulae can occur in such a proof, thus restricting the occurrences of formulae in proofs.

Apart from its usage as a decision procedure for the theorems and non-theorems of a logic, the reason decidability is so important for a logic is that the logic of its meta-theory is then classical. This was argued in Brady [2019], due to each formula of a logic being either provable or not provable in a finite number of steps because of its decidability. Also, the set of its theorems do not overlap with its non-theorems, thus enabling proof as the key concept of a logic to be evaluated in its meta-theory in the usual classical manner.

To construct the normalized natural deduction system for MC, we will closely follow that for the logic DW given in Brady [2006a], "Normalized Natural Deduction Systems for some Relevant Logics I: The Logic DW". The reader will derive some benefit from having this paper on hand. Nevertheless, because of the lapse of time, we will include all the salient parts that are common to this paper, allowing the current paper to be read separately from its [2006a] predecessor, rather than as part of an extended Part II, as was earlier envisaged. The proofs of key results for MC will rely heavily on the corresponding ones in Brady [2006a], but with appropriate alterations and enhancements.

In order to relate their respective natural deduction systems, we note the differences between the logics DW and MC. Apart from the removal of the distribution axiom from DW, MC also contains the Conjunctive Syllogism axiom, $(A \to B)\&(B \to C) \to .A \to C$, and the disjunctive meta-rule, if $A \Rightarrow B$ then $C \vee A \Rightarrow C \vee B$, neither of which is in DW. Whilst the addition of Conjunctive Syllogism will add complexity to the $T \to$ elimination rule in the modified natural deduction system MMC for MC below, the removal of the distribution axiom will simplify the threads and strands within its structures, and the addition of the disjunctive meta-rule will not make a difference as the set of theorems will not change due to the priming property 'if $A \vee B$ is a theorem then A is a theorem or B is a theorem' holding for the logic MC. (See Meyer [1976], Slaney [1984] and [1987], and Brady [2017] for the proof of this property using metavaluations.)

We note here that the normalized natural deduction system should be extendable to include the rules of MC, as well as the theorems of MC, but this will be left for a later time. In this context, the inclusion of the meta-rule MR1 would likely add further rules to the logic MC which can be seen from Brady [1994], where the rules of such a logic with distribution are regular-truth-preserving in comparison to the rules of such a logic without MR1 which are general model structure-validity-preserving. However, for the decidability of the rules, special attention needs to be paid to the deductive subproofs in the corresponding proof of the Corollary to the Subformula Theorem. (See §4 below, where it is shown that any subformula of the formula under test has a depth equal to the depth of subproof in which such a subformula can occur, thus putting a finite limit on the depth of subproofs, i.e. the degree of the formula under test.)

1 The Logic MC

We set out the sentential logic MC, starting with the axioms and rules.

Primitives.

$\sim, \&, \vee, \to$ (connectives)

$p, q, r, ...$ (sentential variables)
Formation Rules.

1. A sentential variable is an atomic formula.
2. If A and B are formulae, then $\sim A, A\&B, A \vee B, A \to B$ are formulae.

Axioms.

1. $A \to A$.
2. $A\&B \to A$.
3. $A\&B \to B$.
4. $(A \to B)\&(A \to C) \to .A \to B\&C$.
5. $A \to A \vee B$.
6. $B \to A \vee B$.
7. $(A \to C)\&(B \to C) \to .A \vee B \to C$.
8. $\sim\sim A \to A$.
9. $A \to \sim B \to .B \to \sim A$.
10. $(A \to B)\&(B \to C) \to .A \to C$.

Rules

1. $A, A \to B \Rightarrow B$.
2. $A, B \Rightarrow A\&B$.
3. $A \to B, C \to D \Rightarrow B \to C \to .A \to D$.

Meta-Rule.

1. If $A \Rightarrow B$ then $C \vee A \Rightarrow C \vee B$.

Note that in MR1 the rule $A \Rightarrow B$ is applied as a single premise rule, as opposed to the two-premise version of the rule, from which distribution in rule-form, $A\&(B \vee C) \Rightarrow (A\&B) \vee (A\&C)$ can be derived. This two-premise rule-form was introduced in Brady [2015] into MC but was rejected again as discussed in Brady [2022], for similar reasons to that for the axiom-form of distribution but applied to a rule setting.

We now proceed to establish the basic normalization, subformula property and decidability for the sentential logic MC of meaning containment.

2 The Modified Natural Deduction System MMC for the Logic MC.

(i) Some Preliminary Definitions.

In Brady [2006a], we facilitated the normalization by first introducing a modified natural deduction system MDW, in place of the Fitch-system FDW given in Brady [1984], so as to align it as close as possible with the final normalized natural deduction system NDW. Here, we start by modifying FDJ for the logic DJ, which is DW + A10, so as to incorporate Conjunctive Syllogism. Importantly, we convert it so that, for each connective, there is just a pair of introduction and a pair of elimination rules, with appropriate conditions attached to some of these rules. These pairs will consist in the T-version and F-version of the rules, where the F-version applies to negative formulae whilst the T-version of the rule applies to positive formulae. (Note that there will be an exception here for the connective $\rightarrow$, which just has the T-version at this stage. An F-version will be introduced later in §3(ii) for the normalization.) This will be facilitated by replacing formulae by signed formulae, TA and FA, within the proofs. (We can think of TA as representing A, and FA as representing $\sim A$.) In order to streamline the normalization process, we also introduce commas and semicolons which separate signed formulae, and which are to be respectively interpreted as conjunction and disjunction. Such a comma will usually be eliminated by a conjunction introduction rule yielding T$A\&B$, and similarly for F$A \vee B$. Such a semicolon will be introduced in the conclusion of the disjunction elimination rule for T$A \vee B$, and similarly for F$A\&B$, replacing the pair of hypotheses introduced for FDJ in Brady [1984] as part of the usual disjunction elimination process. Recall that for this pair of hypotheses, there were added indices, as occurred originally in Anderson [1960] and also in Brady [1984] to represent the set of dependent hypotheses in this case. However, in our case, there are no changes in index sets upon the introduction of a semi-colon via T$A \vee B$ or F$A\&B$. Thus, we define a step as a sequence of signed formulae, separated by commas and semi-colons as recursively introduced below, such that their associativity and commutativity is assumed within their bracketing, and with a single index set, to be used in the natural deduction proofs to follow. A step can also consist of a single signed formula. We will sometimes refer to a step as a proof step. We will define a 'structure' below, which will inductively specify the constituents of a *proof step.*

We proceed with the natural deduction system MMC, in which proofs are set out in the standard Fitch-style, used by Anderson and Belnap in [1975], with the (unmarked) main proof to the left and all its various *subproofs* to its right, the *scopes* of which are usually indicated with vertical lines to the left of their steps with the hypotheses underlined.

We can generally use the term *proof* to apply to either the main proof or any of its subproofs, but we use the term overall proof to apply to the entire proof of a theorem consisting of the main proof and all its subproofs. *Index sets*, consisting of sets of natural numbers, possibly empty, and attached to each step of a proof, will be used to represent the set of hypotheses upon which the proof step depends. (Such hypotheses are entailment hypotheses as opposed to those introduced as a disjunct.) Each subproof has a unique *immediate superproof*, which is the proof that it relates back to, on its immediate left. Generally, for two proofs P and Q, P is a *superproof* of Q iff Q is a *subproof* of P, this meaning that there is a right-to-left chain of immediate superproofs from Q to P. And, P is an *immediate subproof* of Q iff Q is the *immediate superproof* of P. (These are specific subproofs of a particular proof, rather than the general subproofs of the main proof, referred to above. We will use the term subproof, by itself, to refer to subproofs of the main proof.) All theorems under test are eventually proved in the main proof, all steps in which have a null index set as they occur outside the scope of subproofs.

In order to tighten up our natural deduction proofs, we drop the Reiteration Rule of Anderson [1960], by allowing T→E to apply across proofs, i.e. to $\mathrm{T}A \to B$ in a superproof of a proof with TA and TB (or FB and FA) in it. We also drop the Repetition Rule of Anderson [1960], by allowing the hypothesis of a proof to be its conclusion. Also, because of the specific structure of weaker logics such as the logic MC, we are able to structure the index sets so that they have one of the two forms:

> $\emptyset$ or
> a complete subset $\{j, \ldots, k\}$ of the natural numbers, for some natural numbers $k \geq 1$ and $1 \leq j \leq k$, where: a complete subset of the natural numbers is defined as a finite subset of the natural numbers where there are no numerical gaps.

We also need inductive definitions for depth and structure. We define the *depth of a proof* as follows:
The depth of the main proof is 0.
The depth of a subproof is 1 greater than that of its immediate superproof, which can be the main proof.

We define *signs*, *signed formulae*, *opposite sign*, *&-structures*, *∨-structures* and *whole structures* as follows:
A sign is T or F.
If S is a sign and A is a formula, then SA is a signed formula and a structure.
If S is a sign, T or F, then S′ is the opposite sign to that of S, i.e. F or T, respectively.
If $\alpha_1, ..., \alpha_n$ are signed formulae or ∨-structures then $(\alpha_1, ..., \alpha_n)$ is an &-structure and a structure. Associativity and commutativity apply to this sequence of commas, making it an &-multiset.
If $\alpha_1, ..., \alpha_n$ are signed formulae or &-structures then $(\alpha_1; ...; \alpha_n)$ is a ∨-structure and a structure. Associativity and commutativity apply to this sequence of semicolons, making it a ∨-multiset.

Thus, brackets are to be removed internally within a sequence of commas or a sequence of semicolons, but do occur around an entire such sequence. However, we do remove the outside brackets from whole structures, which are structures in their entirety.

As stated above, semicolons are to be understood disjunctively. For example, the structure TA; FB; TC will be subsequently interpreted as $A \vee \sim B \vee C$, in the proof of the Completeness Theorem in §2(v). Commas are to be understood conjunctively, in a similar manner. As mentioned above, we require each signed formula within a whole structure to have the same index set. This is achieved by giving a whole structure a single index set, which then constitutes a proof step, as earlier defined.

To facilitate the addition of A10, Conjunctive Syllogism, we will also make use of substructures, which are introduced recursively as follows for the whole structure β.

1. Any signed formula SA of β is a substructure of β.

2. If α_1 and α_2 are substructures in the same &-multiset of β then α_1, α_2 is a substructure of β, provided α_1, α_2 is not the entire &-multiset and is thus not bracketed.

3. If α_1 and α_2 are substructures in the same $\vee$-multiset of β then $\alpha_1; \alpha_2$ is a substructure of β, provided $\alpha_1; \alpha_2$ is not the entire $\vee$-multiset and is thus not bracketed.

4. Any entire (bracketed) &-multiset of β is a substructure of β.

5. Any entire (bracketed) $\vee$-multiset of β is a substructure of β.

6. The whole structure β is a substructure of β.

In order to structure the emanations in §2(iv) below to deal with A10, any two distinct substructures will either be disjoint or with one properly contained within the other.

(ii) The Rules of MMC.

We now present the rules of MMC, leaving the positioning of signed formulae within proofs until after we define ‘strands of proof’ and ‘threads of proof’ in §2(iii), which are to be understood as sub-subproofs, conjunctively or disjunctively separated from each other within a subproof or the main proof, respectively. It is important to treat strands and threads in such a dual manner because they will need to be interchanged in the proof of the Contraposition Lemma of §3(i). Generally, in the interests of setting out the rules of MMC, we introduce terms with intuitive meanings before giving full definitions later. A key example of the use of these rules will also be given prior to these full definitions in §2(iii) and §2(iv) below.

Hyp. A signed formula of form TH may be introduced as the hypothesis of a new subproof, with a subscript $\{k\}$, where k is the depth of this new subproof.
Any hypothesis thus introduced must subsequently be eliminated by an application of the rule T$\rightarrow$I below.

$T \to I$. From a subproof with conclusion $\mathrm{T}B_a$ on a hypothesis $\mathrm{T}A_{\{k\}}$, infer $\mathrm{T}A \to B_{a-\{k\}}$ in its immediate superproof, where $a = \{j, \ldots, k\}$ and either:

(i) $a - \{k\} = \emptyset$ with $j = k = 1$, or

(ii) $a - \{k\} = \{j, \ldots, k-1\}$ with $k \geq 2$, $1 \leq j \leq k-1$.

The conclusion and hypothesis need not be distinct in T→I(i). The subproof which derives $\mathrm{T}B_a$ from $\mathrm{T}A_{\{k\}}$is called the *subproof of* T→I.
The two cases are as follows:

$$\begin{array}{ll} \left|\begin{array}{l} \underline{\mathrm{T}A_{\{1\}}} \\ \vdots \\ \mathrm{T}B_{\{1\}} \end{array}\right. & \begin{array}{l} \text{Hyp.} \\ \\ \\ \end{array} \\ \mathrm{T}A \to B_{\emptyset} & \mathrm{T}{\to}\mathrm{I(i)} \end{array} \qquad \begin{array}{ll} \left|\begin{array}{l} \underline{\mathrm{T}A_{\{k\}}} \\ \vdots \\ \mathrm{T}B_{a} \end{array}\right. & \begin{array}{l} \text{Hyp.} \\ \\ \\ \end{array} \\ \mathrm{T}A \to B_{a-\{k\}} & \mathrm{T}{\to}\mathrm{I(ii)} \end{array}$$

In both cases, $\mathrm{T}A \to B$ can occur inside a structure. See ‘corresponding thread of proof’ in §2(iii) for its positioning.

$T \to E$. From $\mathrm{T}A_a$ and $\mathrm{T}A \to B_b$, infer $\mathrm{T}B_{a\cup b}$. (Direct version)
From $\mathrm{F}B_a$ and $\mathrm{T}A \to B_b$, infer $\mathrm{F}A_{a\cup b}$. (Contraposed version)
Whilst $\mathrm{T}A_a$ (or $\mathrm{F}B_a$) and its conclusion $\mathrm{T}B_{a\cup b}$ (or $\mathrm{F}A_{a\cup b}$) are located in a proof P, $\mathrm{T}A \to B_b$ is located in the main proof in case (i) below and it is located in P's immediate superproof in cases (ii) and (iii) below.
T→E carries the proviso that one of the following three cases apply:

(i) $b = \emptyset$, in which case $a \cup b = a$, or

(ii) $a = \{k\}$, $k \geq 2$, $b = \{j, \ldots, k-1\}$, $1 \leq j \leq k-1$, in which case $a \cup b = \{j, \ldots, k\}$, or

(iii) $a = \{j, \ldots, k\}$, $k \geq 2$, $b = \{j, \ldots, k-1\}$, $1 \leq j \leq k-1$, in which case $a \cup b = \{j, \ldots, k\} = a$.

We say that T→E is applied to a proof containing $\mathrm{T}A \to B$ into a proof containing $\mathrm{T}A$ (or $\mathrm{F}B$) and $\mathrm{T}B$ (or $\mathrm{F}A$).
The three cases are set out as follows:

$$\begin{array}{l} \mathrm{T} \to \mathrm{E(i)} \\ \\ \\ \mathrm{T}A \to B_{\emptyset} \end{array} \left|\begin{array}{l} \\ \vdots \\ \mathrm{T}A_a \\ \mathrm{T}B_a \end{array}\right. \qquad \begin{array}{l} \mathrm{T} \to \mathrm{E(ii)} \\ \\ \\ \mathrm{T}A \to B_{\{j,\ldots,k-1\}} \end{array} \left|\begin{array}{l} \\ \vdots \\ \mathrm{T}A_{\{k\}} \\ \mathrm{T}B_{\{j,\ldots,k\}} \end{array}\right. \qquad \begin{array}{l} \mathrm{T} \to \mathrm{E(iii)} \\ \\ \\ \mathrm{T}A \to B_{\{j,\ldots,k-1\}} \end{array} \left|\begin{array}{l} \\ \vdots \\ \mathrm{T}A_{\{j,..,k\}} \\ \mathrm{T}B_{\{j,\ldots,k\}} \end{array}\right.$$

[TA can be replaced by FB and TB by FA, for the contraposed version.]
We add T→E(iii) to the two cases (i) and (ii) of Brady [2006a] to enable the Conjunctive Syllogism axiom A10 to be proved. (See §2(v) for a proof of A10.) TA and TB can occur inside a structure in the subproof, but *en bloc* applications of T→E(ii) and T→E(iii) are needed in such a case, as indicated just below.

Such applications of T→E(ii) must be applied *en bloc* (into a proof, when so applied) to all the signed formulae of the whole structure, this requiring a &-multiset of separate entailments for each of these applications, as illustrated in the example below. For T→E(ii), this thereby maintains the commonality of its index set. There is no change in index set for T→E(iii), which is always applied subsequent to a T→E(ii) and is similarly applied *en bloc*, but to a substructure which emanates from a substructure of the whole structure to which T→E(ii) has been applied. Further *en bloc* applications of T→E(iii) can also be made to a substructure, but this still must emanate from a substructure to which T→E(ii) has been applied, as above. (See §2(iv) for the definition of 'emanation of a substructure from a substructure to which T→E(ii) has been applied'.)

These *en bloc* applications all need to be made to both strands and threads, occurring within the emanating substructure. (See §2(iii) for 'strands' and 'threads' of proof, which can be thought of as sub-subproofs within a subproof of the main proof. The strands are separated by commas whilst the threads are separated by semicolons.) The reason for these *en bloc* applications is the failure of Factor principles in the logic MC. Neither the conjunctive nor the disjunctive versions of Factor, viz. $A \to B \to .A\&C \to B\&C$ and $A \to B \to .A\vee C \to B\vee C$, are provable in MC. However, $(A \to B)\&(C \to D) \to .A\&C \to B\&D$ and $(A \to B)\&(C \to D) \to .A \vee C \to B \vee D$ are both derivable in MC, these involving the extra conjunctively added entailment $C \to D$. These two examples would require the *en bloc* applications of T→E(ii) to A, C and $A; C$ respectively.

Note that T→E(ii) must be applied (*en bloc*) only once across all threads and strands in the whole structure, effectively changing the index set. Note also that the index appropriate for the depth of the immediate superproof, i.e., $k-1$, is included in the index set of the conclusion of T→I(ii). However, unlike for T→E(ii), T→E(iii) need not be applied at all into its immediate subproof.

The remaining rules of MMC:

T∼I. From FA_a, infer T∼A_a.
T∼E. From T∼A_a, infer FA_a.
F∼I. From TA_a, infer F∼A_a.
F∼E. From F∼A_a, infer TA_a.
T&I. From TA_a,TB_a, infer T$A\&B_a$.
(TA_a and TB_a can occur anywhere in a &-multiset separated by commas, whilst T$A\&B_a$ can be placed anywhere in the same &-multiset, with TA_a and TB_a removed.)
T&E. From T$A\&B_a$, infer TA_a.
From T$A\&B_a$, infer TB_a.

(If both applied, $\mathrm{T}A_a$ and $\mathrm{T}B_a$ are put into separate strands after applying the rule ,I below. Note that in the process of finding a proof, if one of them introduces an otiose strand of proof, it would thereby be removed.)
$F\&I$. From $\mathrm{F}A_a$, infer $\mathrm{F}A\&B_a$.
From $\mathrm{F}B_a$, infer $\mathrm{F}A\&B_a$.
$F\&E$. From $\mathrm{F}A\&B_a$ infer $\mathrm{F}A_a$;$\mathrm{F}B_a$.
(The conclusion can be placed in an existing ∨-structure. If not, brackets around it are needed.)
$T \vee I$. From $\mathrm{T}A_a$, infer $\mathrm{T}A \vee B_a$.
From $\mathrm{T}B_a$, infer $\mathrm{T}A \vee B_a$.
$T \vee E$. From $\mathrm{T}A \vee B_a$ infer $\mathrm{T}A_a$;$\mathrm{T}B_a$.
(The conclusion can be placed in an existing ∨-structure. If not, brackets around it are needed.)
$F \vee I$. From $\mathrm{F}A_a$,$\mathrm{F}B_a$, infer $\mathrm{F}A \vee B_a$.
($\mathrm{F}A_a$ and $\mathrm{F}B_a$ can occur anywhere in a &-multiset separated by commas, whilst $\mathrm{F}A \vee B_a$ can be placed anywhere in the same &-multiset, with $\mathrm{F}A_a$ and $\mathrm{F}B_a$ removed.)
$F \vee E$. From $\mathrm{F}A \vee B_a$, infer $\mathrm{F}A_a$.
From $\mathrm{F}A \vee B_a$, infer $\mathrm{F}B_a$.
(If both are applied, $\mathrm{F}A_a$ and $\mathrm{F}B_a$. are put into separate strands using ,I below. As for T&E, if one of them introduces an otiose strand of proof, it would thereby be removed.)
$,I$. From $\mathrm{S}A_a$, infer $\mathrm{S}A, \mathrm{S}A_a$. (Again, one needs to see that each $\mathrm{S}A$ in $\mathrm{S}A, \mathrm{S}A_a$ has distinct rules applied to it, so that neither introduces an otiose strand of proof. One must also ensure either that T&I or F∨I is subsequently applied to eliminate this comma or that T→E applications are made to T→-formulae separated by such a comma, which is then eliminated by application of T→I.)
$;E$. From $\mathrm{S}A_a; \mathrm{S}A_a$ infer $\mathrm{S}A_a$.

A formula A is a *theorem of MMC* iff the structure $\mathrm{T}A_\emptyset$ is provable in the main proof.

T→E(ii) and T→E(iii). (An example of some applications.)

$\mathrm{T}A \to B_{\{j,...,k-1\}}$	$\mathrm{T}A \vee C, \mathrm{F}K_{\{k\}}$	
$\mathrm{T}C \to D \vee H_{\{j,...,k-1\}}$	$(\mathrm{T}A; \mathrm{T}C), \mathrm{F}K_{\{k\}}$	$[\mathrm{T} \vee \mathrm{E}]$
$\mathrm{T}K \to M_{\{j,...,k-1\}}$	$(\mathrm{T}B; \mathrm{T}D \vee H), \mathrm{F}M_{\{j,...,k\}}$	$[\mathrm{T} \to \mathrm{E(ii)}]$
$\mathrm{T}B \to G_{\{j,...,k-1\}}$	$(\mathrm{T}B; \mathrm{T}D; \mathrm{T}H), \mathrm{F}M_{\{j,...,k\}}$	$[\mathrm{T} \vee \mathrm{E}]$
$\mathrm{T}D \to X_{\{j,...,k-1\}}$	$(\mathrm{T}G; \mathrm{T}X; \mathrm{T}Z), \mathrm{F}M_{\{j,...,k\}}$	$[\mathrm{T} \to \mathrm{E(iii)}]^2$
$\mathrm{T}H \to Z_{\{j,...,k-1\}}$	$(\mathrm{T}G; \mathrm{T}X \vee Z; \mathrm{T}X \vee Z), \mathrm{F}M_{\{j,...,k\}}$	$[\mathrm{T} \vee \mathrm{I}]^2$
$\mathrm{T}G \to J_{\{j,...,k-1\}}$	$(\mathrm{T}G; \mathrm{T}X \vee Z), \mathrm{F}M_{\{j,...,k\}}$	$[;\mathrm{E}]$
$\mathrm{T}X \vee Z \to W_{\{j,...,k-1\}}$	$(\mathrm{T}J; \mathrm{T}W), \mathrm{F}M_{\{j,...,k\}}$	$[\mathrm{T} \to \mathrm{E(iii)}]^2$
$\mathrm{T}J \vee W \to Y_{\{j,...,k-1\}}$	$(\mathrm{T}J \vee W; \mathrm{T}J \vee W), \mathrm{F}M_{\{j,...,k\}}$	$[\mathrm{T} \vee \mathrm{I}]^2$
	$\mathrm{T}J \vee W, \mathrm{F}M_{\{j,...,k\}}$	$[;\mathrm{E}]$
	$\mathrm{T}Y, \mathrm{F}M_{\{j,...,k\}}$	$[\mathrm{T} \to \mathrm{E(iii)}]$

For the first substructure, $\mathrm{T}B$;$\mathrm{T}D$;$\mathrm{T}H$, to which T→E(iii) is applied, $\mathrm{T}B$ emanates from $\mathrm{T}A$ and $\mathrm{T}D$;$\mathrm{T}H$ emanates from $\mathrm{T}C$, where T→E(ii) has been applied to $\mathrm{T}A$ and $\mathrm{T}C$. Also, for the second substructure $\mathrm{T}G$;$\mathrm{T}X \vee Z$, to which T→E(iii) is applied, $\mathrm{T}G$ emanates from $\mathrm{T}A$ and $\mathrm{T}X \vee Z$ from $\mathrm{T}C$. As indicated, both of these are separate applications of T→E(iii), as they have separate emanations. However, for the third substructure $\mathrm{T}J \vee W$ to which T→E(iii) is applied, $\mathrm{T}J \vee W$ emanates from the substructure $\mathrm{T}A$;$\mathrm{T}C$.

Each of the above entailments with index $\{j, \ldots, k-1\}$, though vertically placed for convenience, are to be placed in separate strands within a thread of proof, to be called 'the corresponding thread of proof' (see below).

(iii) Threads and Strands of Proof.

In order to determine the placement of signed formulae within a proof, we inductively define a *thread of proof* and a *strand of proof* and their associated concepts within the main proof and its subproofs as follows. As indicated above, these are both considered as sub-subproofs within a subproof or the main proof, disjunctively or conjunctively separated, respectively. Also, it is important to maintain duality between these concepts for the purposes of the Contraposition Lemma of §3(i).

Note that there are no extensions to the threads of proof, as introduced in Brady [2006a] for the logic DW, because these are not needed with the absence of the distribution axiom in MC. This allows us to simplify the threads and dualize them with strands in a perspicuous fashion, as indicated below.

(i) The hypothesis of a subproof initiates a new thread of proof, which we will call the basic thread of proof. It also initiates the basic strand of proof, maintaining duality.

(ii) If $\mathrm{T}A \vee B$ (or $\mathrm{F}A \& B$) occurs in a thread or strand of proof and T∨E (or F&E) is applied to it then two adjacent threads of proof are initiated on either side of the introduced comma, within the same structural context as T$A \vee B$ (or F$A\&B$), i.e. with the continuation of the threads and strands in the context. So, if T$A \vee B$ (or F$A\&B$) is in a thread of proof then TA;TB (or FA;FB) replaces it within the $\vee$-structure and if T$A \vee B$ (or F$A\&B$) is in a strand of proof then TA;TB (or FA;FB) replaces it within the &-structure, but bracketed as a pair of threads. Rules can then be applied to the signed formulae on either side of the semicolon, leaving the other side (and, indeed, the rest of the structural context) intact, except of course for *en bloc* applications of T→E(ii) or T→E(iii).

(iii) If TA or FA occurs in a thread or strand of proof and one of the rules T→E (applied to TA or FB from a formula of form T$A \to B$, in all three of the above cases), T∼I, T∼E, F∼I, F∼E, T&E, F&I, T∨I or F∨E is applied to it then the conclusion is still in the same thread or strand of proof, with the same structural context, with the obvious exception for the *en bloc* applications of T→E(ii) and T→E(iii), as indicated above. Also, if T&E or F∨E is applied twice yielding each of its components then these components are put into two adjacent strands, which are initiated by an immediately previous application of ,I. (See (vii) below for ,I.)

(iv) If TA, TB (or FA, FB) occur as separate strands of proof in an &-structure, and T&I (or F∨I) is applied to them then the conclusion T$A\&B$ (or F$A \vee B$) is placed in the &-structure, terminating the two strands.

(v) There are three cases, (I), (II) and (III), for T→I:
(I) The rule T→I(i), where T→E(i) has not been applied into the subproof of the rule T→I(i). In this case, the conclusion of the rule T→I(i) will initiate a strand in the main proof. Where there are more than one such conclusion of the rule T→I(i), they will be separated by commas, forming separate strands.
(II) The rule T→I(ii). The conclusion of T→I(ii) is put in the same thread, in the immediate superproof as all signed formulae of form T$A \to B$ to which T→E(ii) or T→E(iii) is applied into the subproof of T→I(ii), all of which are separated by commas in separate strands. (This thread is called the 'corresponding thread of proof', introduced below.) Specifically, the conclusion of T→I(ii) initiates a strand of proof that immediately follows all the occurrences of signed formulae of form T$A \to B$ to which T→E(ii) or T→E(iii) has been applied. Further, the above strands, separated by commas, are all terminated in this process.

Indeed, each subproof of depth ≥ 2 relates to a specific thread of proof in its immediate superproof. This is called its *corresponding thread of proof*, exemplified below, which must contain all the pertinent T→-formulae concerning the application of the T→I(ii), T→E(ii) and T→E(iii) rules between the subproof and its immediate superproof. These T→-formulae in the corresponding thread of proof are said to form a cluster of T→-formulae or a T→-cluster. Note also that the index sets of all the formulae in a T→-cluster are the same and can either be of the form $\{k\}$ or $\{j, ..., k\}$, for $1 \leq j \leq k-1$. (For the proof of this, see the Index Sets Lemma in §2(v) below.)

Corresponding thread of proof.

$$\begin{array}{l|ll}
 & \mathrm{T}E_{\{k\}} & \\
\cline{2-2}
\mathrm{T}A \to B, \mathrm{T}C \to D, \mathrm{T}B \to J_{\{j,...,k-1\}} \; \cdots & & \\
 & \mathrm{T}A, \mathrm{F}D_{\{k\}} & \\
 & \mathrm{T}B, \mathrm{F}C_{\{j,...,k-1\}} & [\mathrm{T} \to \mathrm{E(ii)}] \\
 & \mathrm{T}J, \mathrm{F}C_{\{j,...,k-1\}} & [\mathrm{T} \to \mathrm{E(iii)}] \\
 & \vdots & \\
 & \mathrm{T}G_{\{j,...,k\}} & \\
\mathrm{T}E \to G_{\{j,...,k-1\}} \;\; [\mathrm{T} \to \mathrm{I(ii)}] & &
\end{array}$$

T→E(iii) applies to the singleton structure TB which emanates from the substructure TA to which T→E(ii) has been applied.
T$A \to B$, T$C \to D$, T$B \to J$ and T$E \to G$ form a T→-cluster and are in the corresponding thread of proof of the subproof with hypothesis TE and conclusion TG.

(III) It remains to consider the application of T→I(i) in the case where T→E(i) has been applied into the subproof of T→I(i). Then, as in (II), the conclusion of T→I(i) occurs immediately after all the occurrences of signed formulae of form T$A \to B$ to which T→E(i) is applied into this subproof, each separated by commas, with the same constraints and definitions as for (II).

(vi) If SA;SA occurs in two threads of proof within the same strand and ;E is applied then these two threads of proof *terminate* at their respective SA, but the conclusion SA of ;E is still contained in the thread or strand of proof prior to the application of the rule T∨E or F&E that introduced the semicolon that was eliminated by ;E, as exemplified below. Quite generally, through the above initiation and termination of threads and strands, the threads and strands of their structural contexts still remain. Thus, in the example below, the threads and strands occurring above such initiation still apply after their termination.

Two threads of proof.

$\mathrm{T}A \vee B_a$

$\mathrm{T}A; \mathrm{T}B_a$

...; ...

$\mathrm{S}D; \mathrm{S}E_c$ (conclusions of $\mathrm{T} \to \mathrm{E(ii)}$, maintaining the threads.)

..., ...

$\mathrm{S}C; \mathrm{S}C_b$

$\mathrm{S}C_b$

The two threads of proof are (i) that which contains TA, SD and SC on the left side and (ii) that which contains TB, SE and SC on the right side. That which contains T$A \vee B$ and the single SC can be in a thread or strand, in accordance with their structural context. Threads (i) and (ii) terminate at their respective SC's. Note that any initiation and termination of adjacent threads of proof occurring between TA and SC on the left and between TB and SC on the right are excluded from this example.

(vii) Two strands of proof are *initiated* by application of the ,I rule, yielding two *adjacent strands* of proof. Two applications of the T&E (or F∨E) rule, i.e. from TA&B, TA&B_a (or F$A \vee B$, F$A \vee B_a$) can be used to infer TA_a and TB_a (or FA_a and FB_a), each of which applies to an adjacent strand of proof. However, we can also apply two rules to the same signed formula, either two introduction rules or an introduction rule and an elimination rule. The only cases of two elimination rules are covered by T&E and F∨E. However, two strands of proof must be terminated in either of two ways. The main way is by an application of T&I (or F∨I), i.e. from TA, TB_a (or, FA, FB_a) we infer TA&B_a (or F$A \vee B_a$), combining the two strands into the one signed formula. It may seem possible to apply two rules within a proof to yield the same signed formula, like the converse of initiation. However, in such a case, one of the strands would become otiose and we would no longer have a standard overall proof (formally introduced below). The other way of terminating separate strands is in the case of a cluster of T→-formulae, upon application of a T→E(ii) or T→E(iii) rule, as above.

(viii) The remaining signed formula occurrences in a thread or strand are repeats of the above introduced signed formulae, which stay the same whilst some rule is being applied to signed formulae somewhere else in the structure. We call these repeats *parametric signed formulae*, whilst the signed formulae to which rules are applied are called *active signed formulae.*

(ix) The conclusion of a proof terminates its basic thread and strand of proof. So, the *basic thread of proof* in a subproof is that which is initiated by the hypothesis of the subproof, does not contain any adjacent threads or strands, continues after they all terminate with the conclusion of their ;E rule, as in (vi) above, and their T&I or F∨I or T→E rules, as in (vii) above, and finally terminates with the conclusion of the subproof. In fact, the basic thread of proof contains just the structures with a single signed formula.

By explicitly including strands, separated by commas, as well as threads, separated by semicolons, we are improving upon Brady [2006a], which only included commas separating threads, with strands formed vertically within threads. This will clarify the strands, which will need to be interchanged with threads in the proof of the Contraposition Lemma of §3(ii). We are also enabling the *en bloc* applications of the T→E(ii) and T→E(iii) rules over both threads and strands together, which will then give clarity to the *last joint application* of the T→E(ii) and T→E(iii) rules applied across a whole structure, which will have to be interchanged with the T→E(ii) rule in the proof of the Contraposition Lemma of §3(ii). This last joint application of the T→E(ii) and T→E(iii) rules will require the use of parametric signed formulae in order to align these T→E(ii) and T→E(iii) applications across a whole structure.

(iv) Emanation of a Substructure from a Substructure to which T→E(ii) has been Applied.

A critical concept associated with that of strands and threads is that of the *emanation of a substructure from a substructure to which T→E(ii) has been applied*. (We will call this substructure to which T→E(ii) has been applied the initial substructure.) So, an application of T→E(iii) is made *en bloc* across a substructure that emanates from a substructure of the whole substructure to which T→E(ii) has been applied, i.e. from the *initial substructure*. A *substructure emanating from an initial substructure* is achieved by first applying T→E(ii) to that substructure of the whole structure, yielding what we will call the *applied substructure*, which has the same structure of commas and semicolons as the initial substructure, but with each of the signed formulae replaced by their consequents of the application of T→E(ii). We can then apply the other derivation rules of MMC (or NMC, later), to the signed formulae, commas and semicolons of the applied substructure to yield an *emanated substructure*, to which T→E(iii) can then be applied. In this process, we refer to the intervening substructures as *emanating substructures*. These derivation rules of MMC can include applications of T→E(iii) itself, with its own emanation process. (We will deal with the F→-rules of NMC in §3 below.) Such emanation follows the strands and threads down a subproof of depth 2 or more, remaining within the confines determined by the initial substructure. So, in particular, if this process reduces to a single signed formula then no comma or semicolon can be subsequently attached to it within the emanation as this would bring in a signed formula from outside such a confinement. Note that T→I(ii) cannot be applied at all to the emanating substructure as its conclusion is then outside the subproof. Note however that a T→-cluster can occur in an emanating substructure with T→E rules applied within the emanation and the T→I(ii) rule applied into the emanation from an immediate subproof.

(v) Lemma and Theorems.

We next prove an important property concerning the changes in index sets, or lack thereof, over the length of a subproof. First, we need the following definition: An overall proof is *standard* when every step is used in the process of deriving the final theorem. Thus, standard overall proofs do not contain any *otiose steps* in their main proof nor in subproofs, i.e. any steps that are not necessary for the derivation of their respective conclusions. Note that we have been deleting such otiose steps in the above for this purpose.

Lemma 1. (The Index Sets Lemma)
(a) Each step in the main proof must have the index set $\emptyset$.
(b) Each step in a subproof of depth 1 must have the index set $\{1\}$.
(c) For any subproof P of depth $k \geq 2$ in a standard overall proof, there is exactly one j in the range, $1 \leq j \leq k-1$, so that each step in P has one of the two index sets, the singleton

set $\{k\}$ and the complete subset $\{j, \ldots, k\}$, the latter index set being obtained by an *en bloc* application of T→E(ii), this being the only rule that can effect a change of index set in the proof P.

Proof. (a) The only rule that can put a signed formula into the main proof is the T→I(i) rule, which introduces the null index set. Then, there is no rule that can change it.

(b) A subproof of depth 1 starts with the index set $\{1\}$, and again there is no rule that can change it.

(c) We start with an outermost subproof P and then apply induction on the number of superproofs in the general direction of the main proof until we reach depth 2. T→I cannot be applied to put its conclusion into the outermost proof. So, T→E(ii) is the only rule that can change an index set within the proof P and is applied from its immediate superproof, *en bloc* across a whole structure at a time. (Note that T→E(iii) does not change the index set within P.) Here, the index set $\{k\}$ is changed to the complete subset $\{j, \ldots, k\}$, where $1 \leq j \leq k-1$, ensuring an index set which includes the index $k-1$ as well as k. Since $\{j, \ldots, k\}$ is not a singleton set, T→E(ii) cannot be reapplied and so no subsequent changes in index set are possible. In a standard overall proof, there can only be one such index set $\{j, .\ldots, k\}$, viz. that of the conclusion of P.

We next let P be a superproof of depth $k \geq 2$. The above argument regarding T→E(ii) still applies here. We need to show that the placing of a conclusion of a T→I(ii) rule into the proof P will not give rise to new index sets. Either T→E(ii) is applied from P into the subproof of the rule T→I(ii) or not.

(α) If T→E(ii) is so applied, the index sets of the subproof of T→I(ii) changes from $\{k+1\}$ to $\{k, k+1\}$ or from $\{k+1\}$ to $\{j, \ldots, k, k+1\}$, yielding a conclusion into P of index set $\{k\}$ or $\{j, \ldots, k\}$, respectively. (By induction hypothesis applied to the subproof of T→I(ii), $\{k, k+1\}$ and $\{j, \ldots, k, k+1\}$ are the only index sets, for some single j such that $1 \leq j \leq k-1$.) So, the conclusion of T→I(ii) will have the same index set as the corresponding $\mathrm{T}A \to B$ formulae to which T→E(ii) were applied. So, T→I(ii) does not give rise to new index sets.

(β) If T→E(ii) is not applied, the index sets of the subproof of T→I(ii) cannot change, which then contradicts the proviso for the T→I(ii) rule. (The T→I(i) rule, in which there is no change in index sets in its subproof, only applies at depth 1.) So, T→E(ii) must be applied and the conclusion of case (α) follows.

We now check to see that all these modifications to the original system FMC do not change its set of theorems. Note that the original system FMC is the natural deduction system for MC which is in the same style of those in Brady [1984]. From this point, we assume that all overall proofs are standard.

Theorem 1. (Soundness Theorem)
If A is a theorem of MC then A is a theorem of MMC.

Proof. We need to check all the axioms and rules of MC. We just give A7, A10 and R3 as examples. Also, see axiom A9 in Brady [2006a]. The meta-rule MR1 need not be checked as the priming property holds for MC, justified by its metacompleteness. (Again, see Meyer [1976], Slaney [1984] and [1987], and Brady [2017] for this.)

A7

1	$\mathrm{T}(A \to C)\&(B \to C))_{\{1\}}$	Hyp.
2	$\mathrm{T}(A \to C)\&(B \to C), \mathrm{T}(A \to C)\&(B \to C)_{\{1\}}$	[,I] (introducing two strands)
3	$\mathrm{T}A \to C, \mathrm{T}B \to C_{\{1\}}$	$[\mathrm{T}\&\mathrm{E}]^2$
4	$\mathrm{T}A \vee B_{\{2\}}$	Hyp.
5	$\mathrm{T}A; \mathrm{T}B_{\{2\}}$	[T∨E]
6	$\mathrm{T}C; \mathrm{T}C_{\{1,2\}}$	[T→E(ii)]
7	$\mathrm{T}C_{\{1,2\}}$	[;E] (terminating the strands)
8	$\mathrm{T}A \vee B \to C_{\{1\}}$.	[T→I(ii)] (completing the T→-cluster)
9	$\mathrm{T}(A \to C)\&(B \to C) \to .A \vee B \to C_{\emptyset}$	[T→I(i)]

A10

1	$\mathrm{T}(A \to B)\&(B \to C)_{\{1\}}$	Hyp.
2	$\mathrm{T}(A \to B)\&(B \to C), \mathrm{T}(A \to B)\&(B \to C)_{\{1\}}$	[,I] (as for A7)
3	$\mathrm{T}A \to B, \mathrm{T}B \to C_{\{1\}}$	$[\mathrm{T}\&\mathrm{E}]^2$
4	$\mathrm{T}A_{\{2\}}$	Hyp.
5	$\mathrm{T}B_{\{1,2\}}$	[T→E(ii)]
6	$\mathrm{T}C_{\{1,2\}}$	[T→E(iii)] (TB emanates from TA)
7	$\mathrm{T}A \to C_{\{1\}}$	[T→I(ii)] (completing the T→-cluster)
8	$\mathrm{T}(A \to B)\&(B \to C) \to .A \to C_{\emptyset}$	[T→I(i)]

R3

1	$\mathrm{T}A \to B_{\emptyset}$	(Assumed to be proved)
2	$\mathrm{T}C \to D_{\emptyset}$	(Assumed to be proved)
3	$\mid$ $\mathrm{T}B \to C_{\{1\}}$	Hyp.
4	$\mid\ \mid$ $\mathrm{T}A_{\{2\}}$	Hyp.
5	$\mid\ \mid$ $\mathrm{T}B_{\{2\}}$	[T→E(i)]
6	$\mid\ \mid$ $\mathrm{T}C_{\{1,2\}}$	[T→E(ii)]
7	$\mid\ \mid$ $\mathrm{T}D_{\{1,2\}}$	[T→E(i)]
8	$\mid$ $\mathrm{T}A \to D_{\{1\}}$	[T→I(ii)]
9	$\mathrm{T}B \to C \to .A \to D_{\emptyset}$	[T→I(i)]

Theorem 2. (Completeness Theorem)
If A is a theorem of MMC then A is a theorem of MC.

Proof. We base the proof on Anderson's method in [1960] for E and EQ, also set out in Brady [1984] for other sentential relevant logics. We introduce quasi-proofs appropriate for MC by adding the following three rules to the list of rules for MMC:

Prefixing. From $\mathrm{T}A \to B_{\emptyset}$, infer $\mathrm{T}C \to A \to .C \to B_{\emptyset}$.

Suffixing. From $\mathrm{T}A \to B_{\emptyset}$, infer $\mathrm{T}B \to C \to .A \to C_{\emptyset}$.

Theorem. $\mathrm{T}A_{\emptyset}$, for any theorem A of MC, may be inserted into the main proof.
These three rules are all applicable in threads and strands of proof within the main proof, the results of which can then be used either in subproofs via T→E(i) or in the main proof via the rules for the other connectives.

As in the method set out in Anderson [1960] and Brady [1984], the completeness theorem is proved by induction on subproofs, starting with an outermost subproof with hypothesis $\mathrm{T}H_{\{k\}}$, replacing its steps by signed formulae in its corresponding thread of proof within its immediate superproof. The resultant overall proof will be a *quasi-proof* appropriate to MC. We continue with this inductive procedure until only the main proof remains, which is then converted into a Hilbert-style proof for MC, applying MR1 to the threads.

Generally, we consider any signed formula $\mathrm{T}A$ or $\mathrm{F}A$, with index set a, within one of the threads or strands of proof within an arbitrary subproof P, where a can be $\{k\}$, with $k \geq 1$, or $\{j, \ldots, k\}$, with $k \geq 2$ and $1 \leq j \leq k-1$. For this procedure, we replace such $\mathrm{T}A$ and $\mathrm{F}A$ by A and $\sim A$, respectively, to form an *unsigned formula.* We next consider an indexed whole structure occurring in the proof P. Such whole structures, when put together to the right of the scope line of the proof, constitute the whole proof in P from hypothesis to conclusion. We define a *whole conjunction-disjunction* as the application of conjunction and disjunction to the signed formulae of a whole structure, in accordance with their respective commas and semi-colons, of all unsigned formulae in the whole structure, bracketed in the same manner as in the structure. We write the whole conjunction-disjunction CD of the whole structure α, containing a particular signed formula TA (or FA) as $\text{CD}_\alpha[(\sim)A]$, the negation preceding A in case the signed formula is FA instead of TA. This whole conjunction-disjunction encapsulates the interpretation of the whole structure as a single formula. For the purposes of T→E(iii) below, we will also apply this terminology to substructures β of a whole structure α, as required for applications of T→E(iii).

As above, a can be either $\{k\}$ or $\{j,\ldots,k\}$, in which case $a-\{k\}$ is either $\emptyset$ or $\{j,...,k-1\}$ in the immediate superproof of P. Note, however, that the immediate superproof, referred to here, is the main proof when $k=1$, and also that *Theorem* is used to replace steps whose index set is $\{k\}$, which also occurs in the main proof. So, for both these simple cases, the immediate superproof is effectively the main proof as $a-\{k\}$ is $\emptyset$ and we will be treating these initially below, as they involve neither T→E(ii) nor T→E(iii), which will then be considered separately.

So, we first consider the case where the structures have the singleton index set $\{k\}$, which applies in both the above simple cases. We then replace each whole structure in turn, which includes a signed formula TA (or FA) in a strand or thread of proof of P, by its *corresponding signed formula*:

$$\text{T}H \to \text{CD}_\alpha[(\sim)A]_\emptyset,\ \text{where T}H \text{ is the hypothesis of the subproof } P.$$

Such corresponding signed formulae with a null index set are placed in suitable strands within the corresponding thread of proof of the main proof.

We determine the quasi-proof replacements for each appropriate rule in turn, given the starting point, $\text{T}H{\to}H_\emptyset$. for the hypothesis T$H$. We initially consider the ∼-rules, T→E(i), T∨I, F&I, T&E, F∨E, the latter two being single applications. In each of these, the form $\text{T}E{\to}G_\emptyset$ is available for some premise E and some conclusion G, from which can be proved $\text{TCD}_\alpha[E]{\to}\text{CD}_\alpha[G]_\emptyset$ by adding respective disjunctions and conjunctions in accordance with the structure α, by repeatedly using theorems of the forms, $\text{T}(C \to C)\&(E \to G) \to .C\&E \to C\&G_\emptyset$ and $\text{T}(C \to C)\&(E \to G) \to .C \vee E \to C \vee G_\emptyset$. Hence, by Prefixing, $\text{T}H \to \text{CD}_\alpha[E] \to .H \to \text{CD}_\alpha[G]_\emptyset$, and then, by T→E(i), $\text{T}H \to \text{CD}_\alpha[E]_\emptyset$ implies $\text{T}H \to \text{CD}_\alpha[G]_\emptyset$ in the main proof. This implication establishes the quasi-proof replacements in the main proof for the above rules in turn.

There are four remaining rules concerning the introduction and elimination of threads and strands:

(I) If there is an introduction of two threads of proof due to T∨E or F&E, then the corresponding signed formulae are unchanged or equivalent due to $\text{T}{\sim}(A\&B) \leftrightarrow {\sim}A \vee {\sim}B_\emptyset$, with the semicolon replaced by ∨ in each case.
(II) If there is an elimination of two threads of proof due to ;E, then the corresponding signed formulae would be equivalent due to $\text{T}A \vee A \leftrightarrow A_\emptyset$.

(III) If there is an introduction of two strands of proof due to ,I, then the corresponding signed formulae would be equivalent due to $\mathrm{T}A\&A \leftrightarrow A_\emptyset$, with the comma replaced by &.
(IV) If there is an elimination of two strands of proof due to T&I or F∨I, then the corresponding signed formulae would be unchanged or equivalent due to $\mathrm{T}{\sim}A\&{\sim}B \leftrightarrow {\sim}(A \vee B)_\emptyset$, with the comma replaced by & in each case.

This completes the available rules that apply to steps in the proof P with index set $\{k\}$. So, $\mathrm{T}H \rightarrow C_\emptyset$ is the corresponding signed formula in the main proof, for a conclusion C of proof P with $k = 1$. This completes what we require for the case $k = 1$. However, for proofs P with $k \geq 2$ and $1 \leq j \leq k-1$, $\mathrm{T}H \rightarrow \mathrm{CD}_\alpha[E_1, ..., E_n]_\emptyset$, is the corresponding signed formula in the main proof with the conjunctions and disjunctions between the signed formulae $E_1, ..., E_n$, from the n strands and threads involved, prior to an application of T→E(ii). As can be seen below, this application would then yield *en bloc* conclusions of the form $G_1, ..., G_n$ from which $\mathrm{TCD}_\alpha[E_1, ..., E_n] \rightarrow \mathrm{CD}_\alpha[G_1, ..., G_n]_{\{j, ..., k-1\}}$ is derivable in the immediate superproof by adding the respective conjunctions and disjunctions by using entailments in the main proof.

For the sake of adding T→E(iii) later, we continue by basing our corresponding signed formulae in the immediate superproof on $\mathrm{TCD}_\alpha[E_1, ..., E_n]$ instead of $\mathrm{T}H$, as this enables the emanated substructures to mesh with the substructures formed from $E_1, ..., E_n$. As for the main proof above, the corresponding signed formulae with non-null index set $\{j, \ldots, k-1\}$ are then placed in suitable strands within the corresponding thread of proof within the immediate superproof Q of P. We will proceed to prove $\mathrm{TCD}_\alpha[E_1, ..., E_n] \rightarrow \mathrm{CD}_\gamma[K_1, ..., K_m]_{\{j, ..., k-1\}}$ for some emanated $\mathrm{CD}_\gamma[K_1, ..., K_m]_{\{j, ..., k-1\}}$ until the final conclusion C of the subproof is reached with corresponding signed formula $TCD_\alpha[E_1, ..., E_n] \rightarrow C_{\{j, ..., k-1\}}$ from which $\mathrm{T}H \rightarrow C_{\{j, ..., k-1\}}$ is derivable by applying Suffixing and T→E(i) to $\mathrm{T}H \rightarrow \mathrm{CD}_\alpha[E_1, ..., E_n]_\emptyset$, which was established above. This will then complete what we require for the remaining case $k \geq 2$ and $1 \leq j \leq k-1$, as it will allow the proof by induction on depth of proof to proceed.

We start by determining the quasi-proof replacements for each appropriate rule in turn, but for the superproof Q. As above, for each of the ∼-rules, T→E(i), T∨I, F&I, T&E, F∨E, the latter two being single applications, and the four remaining rules concerning the introduction and elimination of threads and strands, with reference to the superproof Q, the form $\mathrm{T}J \rightarrow K_\emptyset$ is available for some premise J and some conclusion K. By adding respective conjunctions and disjunctions in accordance with the structure α, $\mathrm{CD}_\alpha[J] \rightarrow \mathrm{CD}_\alpha[K]_\emptyset$. Then, by Prefixing, $\mathrm{TCD}_\alpha[E_1, ..., E_n] \rightarrow \mathrm{CD}_\alpha[J] \rightarrow .\mathrm{TCD}_\alpha[E1, ..., En] \rightarrow \mathrm{CD}_\alpha[K]_\emptyset$ and then, by T→E(i), $\mathrm{TCD}_\alpha[E_1, ..., E_n] \rightarrow \mathrm{CD}_\alpha[J]_{\{j, ..., k-1\}}$ implies $\mathrm{TCD}_\alpha[E_1, ..., E_n] \rightarrow \mathrm{CD}_\alpha[K]_{\{j, ..., k-1\}}$ in the immediate superproof Q of P. This implication then establishes the quasi-proof replacements for the above rules in Q, following an application of T→E(ii), to be treated immediately below.

It remains to consider T→E(ii), T→E(iii) and the T→-clusters. For T→E(ii), the forms $\mathrm{T}E_1 \rightarrow G_{1\{j, ..., k-1\}}, \ldots, \mathrm{T}E_n \rightarrow G_{n\{j, ..., k-1\}}$ are available in the immediate superproof

Q for the application of this rule to the n signed formulae $E_1, ..., E_n$ of the whole structure α. By repeated applications of T&I, $\mathrm{T}(E_1 \to G_1)\&\ldots\&(E_n \to G_n)_{\{j,\ldots,k-1\}}$, from which $\mathrm{TCD}_\alpha[E1, ..., En] \to \mathrm{CD}_\alpha[G_1, ..., G_n]_{\{j,\ldots,k-1\}}$ is derivable by adding the respective conjunctions and disjunctions by using entailments in the main proof. In particular, $\mathrm{T}(A \to B)\&(C \to D) \to .A\&C \to B\&D_\emptyset$ and $\mathrm{T}(A \to B)\&(C \to D) \to .A \vee C \to B \vee D_\emptyset$ are used, which can then be extended from 2 formulae to the n formulae above.

For T→E(iii), the forms $\mathrm{T}J_1 \!\to K_1\,_{\{j,\ldots,k-1\}}, \ldots, \mathrm{T}J_m \to K_m\,_{\{j,\ldots,k-1\}}$ are available in the immediate superproof Q, but applied to an emanated substructure γ with signed formulae $\mathrm{J}_1, ..., J_m$, yielding $K_1, ..., K_m$ respectively with the application of T→E(iii) into the proof P. As above for T→E(ii), $\mathrm{TCD}_\gamma[J_1, ..., J_m] \to \mathrm{CD}_\gamma[K_1, ..., K_m]_{\{j,\ldots,k-1\}}$. Preceding this application of T→E(iii), we must have $\mathrm{TCD}_\beta[B_1, ..., B_p] \to \mathrm{CD}_\gamma[J1, ..., Jm]_{\{j,\ldots,k-1\}}$, where $\mathrm{CD}_\beta[B_1, ..., B_p]$ is the conjunction-disjunction of a substructure β of the whole structure $\alpha[E_1, ..., E_n]$ with signed formulae $B_1, ..., B_p$ from which the substructure γ emanates. This entailment can be seen from the emanation process which initially applies T→E(ii) and then just the ∼-rules, T→E(i), T∨I, F&I, T&E, F∨E, as above, together with the comma and semicolon rules, also as above. Since we have both $\mathrm{TCD}_\beta[B_1, ..., B_p] \to \mathrm{CD}_\gamma[J_1, ..., J_m]_{\{j,\ldots,k-1\}}$ and $\mathrm{TCD}_\gamma[J_1, ..., J_m] \to \mathrm{CD}_\gamma[K_1, ..., K_m]_{\{j,\ldots,k-1|}$, by T&I and Theorem (A10), $\mathrm{TCD}_\beta[B_1, ..., B_p] \to \mathrm{CD}_\gamma[K_1, ..., K_m]_{\{j,\ldots,k-1\}}$. We then replace the individual signed formulae $B_1, ..., B_p$ of $E_1, ..., E_n$ by the conjunction-disjunction $\mathrm{CD}_\beta[B_1, ..., B_p]$, leaving the remainder of $E_1, ..., E_n$ as is. We represent the conjunction-disjunction of this remainder in the form of the outer context CD' so that $\mathrm{CD}'(\mathrm{CD}_\beta[B_1, ..., B_p])$ is just $\mathrm{CD}_\alpha[E_1, ..., E_n]$. Then, we apply the T→E(ii) process as above, but with this replacement, so that the signed formulae in CD' from $E_1, ..., E_n$ yield signed formulae in CD'' from $G_1, ..., G_n$ which correspond exactly to those in the remainder of $E_1, ..., E_n$. And the replacement, $\mathrm{CD}_\beta[B_1, ..., B_p]$ yields $\mathrm{CD}_\gamma[K_1, ..., K_m]$, as above. Putting this result together, $\mathrm{TCD}'(\mathrm{CD}_\beta[B_1, ..., B_p]) \to \mathrm{CD}''(\mathrm{CD}_\gamma[K_1, ..., K_m])_{\{j,\ldots,k-1\}}$, where the outer CD' provides the parametric context for the substructure β within α, as indicated above, and CD'' for the emanated substructure γ within its whole structure, as each rule is treated this same way with common parametric context. So, $\mathrm{TCD}_\alpha[E_1, ..., E_n] \to \mathrm{CD}''(\mathrm{CD}_\gamma[K_1, ..., K_m])_{\{j,\ldots,k-1\}}$ follows by definition. Successive replacements are then used for subsequent applications of T→E(iii), with the preceding replacements remaining in place for the treatment of later applications, which will apply to substructures containing any earlier substructures. Subsequent applications of T→E(iii) can also be made to disjoint substructures, which would mean that their corresponding replacements would have to be included in the remainders of $E_1, ..., E_n$ in the same way as the above replacements have been included within α.

Lastly, we examine T→I(ii), in conjunction with the T→E(ii) and T→E(iii) rules, forming a T→-cluster. This consists of a series of T→E(ii) and T→E(iii) rules, applied to T→-formulae, separated by commas, with index set $\{j, \ldots, k-1\}$ in the superproof Q, followed by a single concluding T→I(ii) rule, yielding the entailment from the hypothesis of P to its conclusion, also with index set $\{j, \ldots, k-1\}$. What we will show is that the conclusion of the T→I(ii) rule is derivable from the conjunction of the entailments to which the T→E(ii) and T→E(iii) rules are applied. This derivation will be conducted in a special subproof so that their respective entailment can be established.

To do this, we let $\mathrm{T}A_1 \to B_1, ..., \mathrm{T}A_e \to B_e$ be the entailments to which T→E(ii) and T→E(iii) are applied in Q, all conjoined but with index $\{1\}$. We take this conjunction to be the hypothesis of Q', taken from these key elements of Q. We then apply T&E to eliminate these conjunctions. Let $\mathrm{T}C_{\{2\}}$ be the hypothesis of the proof P' with index set $\{2\}$, which can only be impacted from outside of itself by the T→-rules, applied from the main proof, in the case of T→E(i), from Q' in the case of the T→E(ii) and T→E(iii), and from a subproof of P', in the case of T→I(ii). We let $\mathrm{T}D_{\{1,2\}}$ be the conclusion of P', derived as per the proof P, but with indices $\{1\}$ and $\{1,2\}$, using the various T→-rules. Then, by T→I(ii), $\mathrm{T}C \to D_{\{1\}}$ is derived in Q' as the conclusion of the T-cluster. Finally, apply T→I(i) to yield $\mathrm{T}(A_1 \to B_1)\&...\&(A_e \to B_e) \to .C \to D_\emptyset$. Hence, by *Prefixing*, $\mathrm{TCD}_\alpha[E_1, ..., E_n] \to (A_1 \to B_1)\&...\&(A_e \to B_e) \to .\mathrm{CD}_\alpha[E_1, ..., E_n] \to .C \to D_\emptyset$ and $\mathrm{TCD}_\alpha[E_1, ..., E_n] \to .C \to D_{\{j,...k-1\}}$ can be proved, as required in Q, using the induction assumption $\mathrm{TCD}_\alpha[E_1, ..., E_n] \to (A_1 \to B_1)\&...\&(A_e \to B_e)_{\{j,...k-1\}}$.

At the end of the induction on subproofs, we have all the corresponding signed formulae with a null index set placed into the basic thread of proof of the main proof. We just need to show how this basic thread of proof can be converted into a Hilbert proof within MC. We replace, in the main proof, steps of form $\mathrm{T}A$ by A and $\mathrm{F}A$ by $\sim A$, all commas are replaced by '&', and all semicolons are replaced by '∨', maintaining the brackets. The rules of MMC readily convert into rules and theorems of MC, as do the additional rules of *Prefixing, Suffixing* and *Theorem*. However, we do need to consider the application of the meta-rule MR1, which is a rule applied into a disjunctive context. So, if $A \Rightarrow B$ then $C \vee A \Rightarrow C \vee B$ and, together with $C\&A \Rightarrow C\&B$, $\mathrm{CD}_\alpha[A] \Rightarrow \mathrm{CD}_\alpha[B]$ follows. So, any theorem A of MMC is provable in MC.

3 The Normalized Natural Deduction System NMC.

In natural deduction for classical logic, there is usually a plethora of rules allowing substitution of equivalents as well as a range of deductive rules, and this seems to capture the natural freedom of deduction in classical logic. For relevant logic, one has to be more careful to ensure that one stays within the confines of a particular relevant logic. If one considers how one would normally perform a Fitch-style natural deduction in relevant logic, there is a fairly tight routine in the methodology with each connective generating certain rules and procedures, enabling a fairly quick proof of a theorem. Indeed, there is a difference between natural deduction, as one would normally perform it for relevant logics, and the breadth allowed in the systems set out in Brady [1984] or even that in §2 above for MC. The feature of the systems in [1984] and §2 is that they are broad enough to follow the Hilbert-style proof procedure, using Modus Ponens and Affixing Rules, as occurs in the proof of soundness in Theorem 1. What normalization will do is to fix the system so that one is forced to apply a tight methodology, divorcing the system from the Hilbert-style approach.

We give a striking example of the MC theorem $A\&B \to B \vee C$, proved in accordance with the Hilbert-style soundness argument, as against a normalized proof.

Soundness Argument		Normalized Argument
$\mathrm{T}A\&B_{\{1\}}$		$\mathrm{T}A\&B_{\{1\}}$
$\mathrm{T}B_{\{1\}}$		$\mathrm{T}B_{\{1\}}$
$\mathrm{T}A\&B \to B_{\emptyset}$	[T→I(i)]	$\mathrm{T}B \vee C_{\{1\}}$
$\mathrm{T}B_{\{1\}}$		$\mathrm{T}A\&B \to B \vee C_{\emptyset}$
$\mathrm{T}B \vee C_{\{1\}}$		
$\mathrm{T}B \to B \vee C_{\emptyset}$	[T→I(i)]	
$\mathrm{T}B \to B_{\{1\}}$		
$\mathrm{T}A\&B_{\{2\}}$		
$\mathrm{T}B_{\{2\}}$	[T→E(i)]	
$\mathrm{T}B_{\{1,2\}}$	[T→E(ii)]	
$\mathrm{T}B \vee C_{\{1,2\}}$	[T→E(i)]	
$\mathrm{T}A\&B \to B \vee C_{\{1\}}$		
$\mathrm{T}B \to B \to .A\&B \to B \vee C_{\emptyset}$	[T→I(i)]	
$\mathrm{T}B_{\{1\}}$		
$\mathrm{T}B \to B_{\emptyset}$	[T→I(i)]	
$\mathrm{T}A\&B \to B \vee C_{\emptyset}$	[T→E(i)]	

The Soundness Argument uses the three axioms A3, A5 and A1, together with the Affixing Rule (R3) and Modus Ponens (R1). One should note the introduction of T→, where the four marked T→I(i) rules occur, and the subsequent elimination of T→, where the three marked T→E(i) rules and the marked T→E(ii) rule occur. Indeed, when such introductions and eliminations are removed in accordance with our normalization procedure below, we will get the Normalized Argument on the right, which illustrates the tight routine normally used in relevant natural deduction. Note that, in its subproof, we start with a hypothesis, eliminate a connective &, reaching a suitable turning point $\mathrm{T}B_{\{1\}}$, and introduce a connective ∨, reaching the conclusion of the subproof. The final conclusion, $\mathrm{T}A\&B \to B \vee C_{\emptyset}$, initiates the basic thread of proof of the main proof, which is as far as it goes. The elimination rule followed by an introduction rule in the subproof goes some way towards illustrating the structure of normalized proof.

(i) The Structure of Normalized Proof

We proceed to determine what the structure of normalized overall proofs would look like. To normalize MMC we need to remove deviations in the overall proof, where such deviations are the introduction of a connective and the subsequent elimination of it from the same signed formula. As we have seen above, this would also include the introduction and elimination of T→-formulae, involving more than one proof. In order to do all this, we remove *IE-turning points*, which are signed formulae, introduced by a connective introduction rule and subsequently eliminated by the elimination rule for the same connective.

Once these deviations are removed, as will be shown in Theorem 3, each proof will be made up of some *normal pieces of proof (n.p.p.)*, which begin at a whole structure or substructure which is a *starting point*, eliminate connectives in turn (except →) until we reach a whole structure or emanating substructure which is an EI-turning point, and then introduce connectives in turn (except →) until we reach a whole structure or emanating substructure of the starting point which is a *finishing point*. The sequence of eliminations is called an *elimination phase of the n.p.p.* and the sequence of introductions is called an *introduction phase of the n.p.p.*

We proceed to clarify the above concepts. Firstly, there may be no elimination phase or no introduction phase. Indeed, there may be neither, in which case the entire n.p.p. reduces to a single step in the proof. In such a case, we continue to apply the turning point terminology. The hypothesis of a subproof is always a single starting point for the first n.p.p. and the conclusion of a proof is always a single finishing point for the last n.p.p.

EI-turning points (EI-t.p.'s) occur in single strands and threads of proof but, by use of parametric signed formulae, they can be made to line up into whole structures or emanating substructures of the starting structure, called *EI-turning structures (EI-t.s.'s)*. This is done as follows. Once an EI-t.p. is reached in one strand or thread of proof it is repeated until all the other strands and threads in the structure are EI-t.p.'s, creating a single whole structure or emanating substructure.

EI-t.p.'s may also consist of a *cluster of T→-formulae*, as follows. A cluster of T→-formulae consists of T→-formulae in a thread to which T→E(ii) or T→E(iii) rules are applied into an immediate subproof, together with the T→-formula which is the conclusion of T→I(ii) from this subproof. The thread of proof involved is the corresponding thread of proof for the immediate subproof. In such a cluster, a number of strands, called *initial strands*, within a single thread can terminate in T→-formulae, to which T→E(ii) or T→E(iii) is applied into an immediate subproof, yielding a single T→-formula, called the *latter T→-formula*, upon subsequent application of T→I(ii), thus eliminating the strands.

The remaining starting and finishing points of n.p.p.'s are determined by the application of T→E(ii) or T→E(iii) from the immediately preceding subproof into this subproof with the TA (or FB) of the rule being a finishing point of an n.p.p. and the TB (or FA) of the rule being the starting point of another n.p.p.. Due to the *en bloc* application of T→E(ii) to a whole structure and T→E(iii) to an emanating substructure, all the corresponding finishing points can be parametrically put into the same structure, called a *finishing structure*, and similarly all the corresponding starting points can be parametrically put in the same structure, called a *starting structure*.

Applications of ,I generally occur in the elimination phase of an n.p.p., as their introduction enables one to eliminate &, whilst the applications of ;E generally occur in the introduction phase of an n.p.p., as their elimination enables one to introduce ∨.

Now for the lemma which determines where n.p.p.'s can occur in a normal proof.

Lemma 2. (The N.P.P. Lemma)
(a) For the main proof, there is only one n.p.p. with introduction rules only. This means that T→E(i) is removed, as well as T∨E and F&E which introduce semicolons into the main proof, together with T&E and F∨E which introduce commas into the main proof. However, commas will still be introduced upon multiple applications of T→I(i), which introduce separate strands for each of its concluding T→-formulae. So, each step consists of single signed formulae separated by commas, each of which are to be eliminated by a T&I or F∨I rule, in the process of deriving the concluding formula of the system.

(b) For proofs of depth 1, there is only one n.p.p.

(c) For proofs of depth 2 or greater, there are two or more n.p.p.'s, one which finishes by applying T→E(ii), all structures being with the index set $\{k\}$, one which finishes with the conclusion of the subproof, and ones starting with the conclusion of the application of T→E(ii) or an application of T→E(iii) and finishing with an application of T→E(iii) into a emanating substructure, these additional n.p.p.'s being all with index set $\{j, ..., k\}$. [Similarly, for the F→-rules, to be introduced in (ii) of the section.]

Proof.
(a) The only way to start a second n.p.p. is by application of a T→E(i) rule. Since T→I(i) is the only rule which puts a signed formula of form T$A \to B$ (or, indeed, any signed formula) into the main proof, any subsequent rule can only introduce connectives as we cannot eliminate T$A \to B$ in a normalized proof. Thus, T→E(i) cannot be applied from the main proof, nor can any elimination rule.

(b) As for (a), the only way to start a second n.p.p. is by application of T→E(i), (ii) or (iii), which respectively either does not apply or does not apply into a subproof of depth 1.

(c) Due to the Index Sets Lemma, there are exactly two index sets in the subproof. As it is not possible to change an index set within an n.p.p., there are at least two n.p.p.'s established through the application of T→E(ii) and possibly T→E(iii). It is not possible to re-apply T→E(ii) once it is applied, but there can be multiple applications of T→E(iii), each creating starting points for an n.p.p. So, there are two n.p.p.'s, with additional n.p.p.'s being created for each initial and each subsequent application of T→E(iii).

The N.P.P.'s in a Normalized Proof of the Theorem:

$$(A \to B\&E)\&(C\&D \to C) \to .A \vee (C\&D) \to B \vee C.$$

$\quad \mathrm{T}(A \to B\&E)\&(C\&D \to C)_{\{1\}}$	
$\quad \mathrm{T}A \to B\&E, \mathrm{T}C\&D \to C_{\{1\}}$	by ,I and [T&E]^2
$\quad\quad \mathrm{T}(A \vee (C\&D)_{\{2\}}$	
$\quad\quad \mathrm{T}A; \mathrm{T}C\&D_{\{2\}}$	
$\quad\quad \mathrm{T}B\&E; \mathrm{T}C_{\{1,2\}}$	
$\quad\quad \mathrm{T}B; \mathrm{T}C_{\{1,2\}}$	
$\quad\quad \mathrm{T}B \vee C; \mathrm{T}B \vee C_{\{1,2\}}$	by [T$\vee I$]^2
$\quad\quad \mathrm{T}B \vee C_{\{1,2\}}$	
$\quad \mathrm{T}A \vee (C\&D) \to B \vee C_{\{1\}}$	
$\mathrm{T}(A \to B\&E)\&(C\&D \to C) \to .A \vee (C\&D) \to B \vee C_{\emptyset}$	

(a) The main proof has a single signed formula n.p.p., i.e. the final theorem.

(b) The first subproof has one n.p.p., starting with the hypothesis and an elimination of &, followed by the conclusion of the subproof which is an EI-turning point. Note the absence of introduction rules in this n.p.p.

(c) The second subproof has two n.p.p.'s. The top n.p.p. starts with the hypothesis, eliminates the disjunction, yielding the EI-turning structure TA;T$C\&D$, which is also the finishing structure. T→E(ii) is then applied to start the second n.p.p. whereupon the & is eliminated yielding the EI-turning point TB;TC. Disjunction is then introduced, reaching the finishing point T$B \vee C$ via an application of ;E.

(ii) Preparing for Normalization.

To continue the normalization process, we need to introduce a new pair of rules. The new rules, F→I and F→E, given below, are needed to prove the contraposition lemma, essential for the proof of normalization. Indeed, they are the contraposed forms of the respective T→E and T→I rules. We call this expanded natural deduction system *NMC*, for the normalized system.

F→I. From a derivation of T$B_{a\cup b}$ from TA_a or F$A_{a\cup b}$ from FB_a, occurring within a subproof, infer F$A \to B_b$ in the corresponding strand of proof in its immediate superproof. [Think of the F→I rule as a contraposed T→E rule, but applied into its corresponding strand of proof in an immediate subproof.]

We code the signed formula $\mathrm{T}B_{a\cup b}$ or $\mathrm{F}A_{a\cup b}$ with F→I.
All the signed formulae $\mathrm{F}A \to B_b$, introduced by F→I from the subproof, will form a structure, separated by semicolons, with this structure lying in the corresponding strand of proof of the subproof. These F→-formulae will also be preceded in this strand by another F→-formula, eliminated (as below) by F→E. (These F→-formulae then form a cluster in a similar manner to the clusters of T→-formulae, defined in §3(i).) Such signed formulae, when there is more than one, will initiate new strands of proof.

We call this immediate subproof an *F-subproof*, as distinct from our usual proofs into which T→E(ii) and T→E(iii) rules are applied, which we will now call *T-subproofs*.

We break F→I up into two cases, aligning with those of T→E(ii) and T→E(iii):

(ii) $a = \{k\}$, $k \geq 2$, $1 \leq j \leq k-1$, $b = \{j, \ldots, k-1\}$ and $a \cup b = \{j, \ldots, k\}$.

(iii) $a = \{j, \ldots, k\}$, $k \geq 2$, , $1 \leq j \leq k-1$, $b = \{j, \ldots, k-1\}$ and $a \cup b = a = \{j, \ldots, k\}$.

Just like T→E(ii) and T→E(iii), F→I(ii) and F→I(iii) must be applied *en bloc* to the signed formulae of, respectively, a whole structure or an emanation of a substructure, maintaining its common index set.
Note that a case (i), corresponding to T→E(i), is not needed as the F→-rules are introduced as contrapositives to prove the following Contraposition Lemma, which is applied in the proof of the Normalization Theorem in section (iii) to a subproof. Thus, F→-formulae can only occur in subproofs and hence there are no F→-formulae in the main proof.

An F→I(ii) rule, applied *en bloc* to a 2-element structure.

$$
\begin{array}{l|l}
 & . \\
 & \mathrm{T}C, \mathrm{T}E_{\{k\}} \\
 & TD, \mathrm{T}G_a \qquad\qquad [\mathrm{F}{\to}\mathrm{I(ii)}] \\
 & . \\
\end{array}
$$

$$
\begin{array}{l}
\mathrm{F}C \to D; \mathrm{F}E \to G_{a-\{k\}} \\
\vdots
\end{array}
$$

Here, we put $\{k\}$ for a, $a - \{k\}$ for b, and a for $a \cup b$, in F→I.

F→E. From $\mathrm{F}A \to B_{a-\{k\}}$, infer an F-subproof with conclusion $\mathrm{T}B_a$ on a hypothesis $\mathrm{T}A_{\{k\}}$, with $\mathrm{F}A \to B_{a-\{k\}}$ occurring in its corresponding strand of proof, where:

(ii) $a = \{j, \ldots, k\}$, $k \geq 2$, $1 \leq j \leq k-1$, and hence $a - \{k\} = \{j, \ldots, k-1\}$.

Note that there is no case (i) corresponding to that for T→I(i) as there are no F→-formulae in the main proof.

[Think of the rule as a contraposed T→I rule, but applied into an immediate subproof from its corresponding strand of proof.]

We code the structure $\mathrm{F}C \to D, \ldots_b$ (from the F→I rule) with F→E, as it is not until the F-subproof is concluded, with its associated structure $\mathrm{F}C \to D, \ldots_b$, that we can say that $\mathrm{F}A \to B_{a-\{k\}}$ is eliminated. Also, $\mathrm{F}C \to D, \ldots_b$ is the next structure after $\mathrm{F}A \to B_{a-\{k\}}$ in its strand of proof. Here, b will be $a - \{k\}$, as in the diagram below.

The F→E rule.

$\mathrm{F}A \to B_{a-\{k\}}$

| $\mathrm{T}A_{\{k\}}$

| . .

| .

| $\mathrm{T}B_a$

$\mathrm{F}C \to D_{a-\{k\}}$ [F→E]

The two F→-rules in tandem.

.

$\mathrm{F}A \to B_{a-\{k\}}$

| $\mathrm{T}A_{\{k\}}$

| . .

| $\mathrm{T}C, \mathrm{T}E, \mathrm{F}J_{\{k\}}$

| $\mathrm{T}D, \mathrm{T}G, \mathrm{F}H_a$ [F→I]

| .

| $\mathrm{T}B_a$

$\mathrm{F}C \to D; \mathrm{F}E \to G; \mathrm{F}H \to J_{a-\{k\}}$ [F→E]

Again, for F→I, we put $\{k\}$ for a, $a - \{k\}$ for b and a for $a \cup b$.

As we can see, F→I and F→E work in tandem, using the same F-subproof and its corresponding strand of proof, and, in particular, F→E cannot be applied without F→I also being applied into the same F-subproof. Also, it is the T→-rules that apply into T-subproofs and the F→-rules that apply into F-subproofs; there is no mixing of T→- and F→-rules, with the exception of T→E(i) which can apply into both T- and F-subproofs. As with T-subproofs, the hypothesis TA of an F-subproof initiates a new thread or strand of proof.

Before we go on, we expand on the notion of a deviation. We also need to count as deviations introductions and eliminations of a connective around ,I and ;E rule applications, involving the three associated strands or threads of proof. Indeed, we need to remove these and all the above deviations to establish a normal proof and to ensure, for the proof of the subformula property to follow (see Theorem 4), that each formula in the overall proof is a subformula of the final theorem.

Further, there are some complications where applications of the ,I and ;E rules occur in between the introduction and elimination rules of an IE-turning point. The penultimate signed formulae of the adjacent strands or threads of proof, prior to such a ,I or ;E rule, could be the same or different, according to the particular introduction rule used, or one could even be an introduction rule and the other an elimination rule. This latter case will be specially considered in the proof of our normalization theorem (Theorem 3).

In the former case, where only introduction rules are applied prior to ,I or ;E, the $\sim$-rules are fairly straightforward, but $\to$-, &- and $\vee$-rules need some consideration, these being attended to in the proof of Theorem 3 below.

In readiness for normalization, we next prove a lemma showing that one can contrapose an entire subproof.

Lemma 3. (Contraposition Lemma)
In a standard overall proof of NMC, any subproof S with $\mathrm{T}B_a$ derived from $\mathrm{T}A_{\{k\}}$ can be contraposed to form a derivation S' of $\mathrm{F}A_a$ from $\mathrm{F}B_{\{k\}}$.

Proof. We replace each rule by its contraposed form, starting at the conclusion $\mathrm{T}B_a$ and working up the subproof until the premise $\mathrm{T}A_{\{k\}}$ is reached. We replace the index sets a by $\{k\}$, up to the last joint application of T$\to$E(ii) and T$\to$E(iii) across a whole structure, and index sets $\{k\}$ by a, leaving the remainder unchanged, for subproofs of depth 2 and higher. (Here, the last applications of T$\to$E(iii) become absorbed into applications of T$\to$E(ii) and any earlier residual application of T$\to$E(ii) would become an application of T$\to$E(iii). Emanations would just go in the opposite direction.) Clearly, indices in proofs of depth 1 are unchanged. Any subproofs of S remain as they are, except for a possible change of indices (see below). We contrapose each connective rule of S by keeping the connective as is and changing its sign, T to F or F to T, and changing the type of rule from elimination to introduction or from introduction to elimination. That is, T$\sim$I becomes F$\sim$E, F$\vee$E becomes T$\vee$I, etc.

In order to replace T$\vee$E by F$\vee$I, we see that we need to replace a pair TA;TB, separated by a semicolon, by FA,FB, separated by a comma, so as to yield F$A \vee B$ by F$\vee$I. Similarly, F&E is replaced by T&I. Conversely, to replace T&I by F&E, we need to replace the signed formulae TA,TB, separated by a comma, by the pair FA;FB, separated by a semicolon. Similarly, F$\vee$I is replaced by T$\vee$E. Thus, we also need to interchange the commas and semicolons as well, which means interchanging threads and strands. So, two adjacent threads of proof are replaced by two adjacent strands of proof, both with the same bracketing, and vice versa.

Replacing $\mathrm{T}\vee\mathrm{E}$ by $\mathrm{F}\vee\mathrm{I}$

$$\left|\begin{array}{l} \mathrm{T}A \vee B_a \\ (\mathrm{T}A;\mathrm{T}B)_a \end{array}\right| \begin{array}{l} (\mathrm{F}A,\mathrm{F}B)_{\{k\}} \\ \mathrm{F}A\vee B_{\{k\}} \end{array}$$

Replacing T&I by F&E. (Both with index change)

$$\left|\begin{array}{l} (\mathrm{T}A,\mathrm{T}B)_a \\ \mathrm{T}A\&B_a \end{array}\right| \begin{array}{l} \mathrm{F}A\&B_{\{k\}} \\ (\mathrm{F}A;\mathrm{F}B)_{\{k\}} \end{array}$$

These adjacent threads or strands of proof may need to be adjusted (by addition of parametric signed formulae) to ensure that any application of T→E(ii) or F→I(ii) occurs *en bloc* across the whole structure and any application of T→E(iii) or F→I(iii) occurs *en bloc* across the appropriate emanation of a substructure to which T→E(ii) or F→I(ii) has been applied.

To continue the replacement, the ;E rule would be replaced by the ,I rule, and vice versa. Again, the commas and semicolons are interchanged.

Let us consider the →-formulae in some detail. First, we replace T→E(ii) by F→I(ii), and conversely, with appropriate change of indices and with respect to a common subproof, which is a T-subproof or an F-subproof, appropriately. This involves replacing a structure consisting of $\mathrm{T}C \to D$'s, separated by commas and bracketed accordingly, to which T→E(ii) is applied (thus terminating the respective strands), by a structure consisting of $\mathrm{F}C \to D$'s, separated by semicolons and bracketed accordingly, to which F→I(ii) is applied (thus initiating respective threads). The converse is just the reverse of this procedure. If there is a change of indices required, i.e. from $\{k\}$ to $\{j,\ldots,k\}$ or conversely, a corresponding change is made to the T- or F-subproof, i.e. from $\{k,k+1\}$ to $\{j,\ldots,k,k+1\}$, or conversely. This change would then be propagated in turn through its following subproofs.

Replacing T→E(ii) by F→I(ii), with index set $\{j,\ldots,k\}$ replaced by $\{k\}$.

$$\begin{array}{ll} \mathrm{T}A \to B, \mathrm{T}C \to D_{\{j,\ldots,k\}} & \\ . & \left| \begin{array}{l} \mathrm{T}A, \mathrm{T}C_{\{k+1\}} \\ \mathrm{T}B, \mathrm{T}D_{\{j,\ldots,k+1\}} \end{array}\right. \end{array} \qquad \begin{array}{ll} . & \left| \begin{array}{l} \mathrm{T}A, \mathrm{T}C_{\{k+1\}} \\ \mathrm{T}B, \mathrm{T}D_{\{k,k+1\}} \end{array}\right. \\ \mathrm{F}A \to B; \mathrm{F}C \to D_{\{k\}} & \end{array}$$

Further, T→E(iii) is replaced by F→I(iii) and conversely, in a similar manner as the replacement of T→E(ii) by F→I(ii) and conversely, but with index set $\{k+1\}$ of $\mathrm{T}A$,$\mathrm{T}C$ changed to $\{j,\ldots,k+1\}$ and $\{k,k+1\}$, respectively.

Secondly, we replace T→I(ii) by F→E(ii) and conversely. Here, we replace $\mathrm{T}A \to B$, introduced by T→I(ii), by $\mathrm{F}A \to B$, eliminated by F→E(ii), and conversely. Again, the common subproof is a T- or F-subproof, respectively.

Replacing T→I(ii) by F→E(ii), with index set $\{k\}$ replaced by $\{j,\ldots,k\}$.

$$\begin{array}{l|l} \mathrm{T}A_{\{k+1\}} \\ \hline . \\ \mathrm{T}B_{\{k,k+1\}} \end{array} \qquad\qquad \mathrm{F}A \rightarrow B_{\{j,\ldots,k\}}$$

$$\mathrm{T}A \rightarrow B_{\{k\}} \qquad\qquad \begin{array}{|l} \mathrm{T}A_{\{k+1\}} \\ \hline . \\ \mathrm{T}B_{\{j,\ldots,k+1\}} \end{array}$$

We also need to consider applications of T→E and F→I into the subproof S. Since T→E(ii) derives $\mathrm{T}D_a$ from $\mathrm{T}C_{\{k\}}$ or $\mathrm{F}C_a$ from $\mathrm{F}D_{\{k\}}$, we just replace one of these by the other, with the index sets being changed as required. F→I(ii) is similar. Further, T→E(i) derives $\mathrm{T}D_c$ from $\mathrm{T}C_c$ or $\mathrm{F}C_c$ from $\mathrm{F}D_c$, where c is either a or $\{k\}$. We replace one of these by the other, changing both index sets as required.

(iii) The Normalization Theorem.

We can now prove the normalization result for NMC.

Theorem 3. (Normalization Theorem)
A is a theorem of MMC iff A is a theorem of a normalized NMC.

Proof.
L⇒R. (The theorem is expressed as a two-way rule.)
(i) We start by eliminating pairs of introduction and elimination rules for each of the connectives in turn, so as to eliminate the IE-turning points.
→.

<u>T → .</u>

$$
\begin{array}{ll}
\left|\begin{array}{l} \underline{\mathrm{T}A_{\{k\}}} \\ . \\ . \\ \mathrm{T}B_a \end{array}\right. & \\
...,\mathrm{T}A \to B, ..._{a-\{k\}} & [\mathrm{T}{\to}\mathrm{I}] \\
\left|\begin{array}{l} . \\ ..., \mathrm{T}A(\text{or } \mathrm{F}B), ..._b \\ ..., \mathrm{T}B(\text{or } \mathrm{F}A), ..._c \\ . \\ . \end{array}\right. & [\mathrm{T}{\to}\mathrm{E}]
\end{array}
$$

To eliminate T$A \to B$ from its thread of proof, we place the top proof of TB_a from T$A_{\{k\}}$ into the space between TA (or FB), with index b and TB (or FA), with index c, within the parametric threads and strands of proof in the bottom proof below. Any use of FB and FA invokes the Contraposition Lemma.

The *en bloc* application of T→E(ii) in the top proof is aligned in the bottom proof so that an overall *en bloc* application of T→E(ii) in the bottom proof can include such an *en bloc* application from the top proof. As required, any applications of T→E(iii) in the top proof are made in the bottom proof *en bloc* across the same substructure emanating from the same substructure to which the top T→E(ii) was applied but now within the bottom proof.

However, in the event that T→E(iii) is the form of T→E of the bottom proof, the *en bloc* application of T→E(ii) in the top proof is aligned in the bottom proof so that the *en bloc* application of T→E(iii) in the bottom proof can incorporate such an *en bloc* application of T→E(ii) from the top proof. Any applications of T→E(iii) in the top proof are then made in the bottom proof *en bloc* across the same substructure but with the emanation traced back through the TA to the application of T→E(ii) in the bottom proof.

We make the following adjustments to the indices. Three cases arise:

(I) $a = \{k\}$, $k = 1$, $a - \{k\} = \emptyset$, and so $b = c$, with T→E(i) applied.

(α) Let b be non-null. Replace the index sets $\{1\}$ in the top proof by b $(= c)$ and the other individual indices $1 + n$ $(n \geq 1)$ by $max(b) + n$ throughout the subproofs of the top proof. We just need to spell out the index changes for the →-rules.
[We call a rule application *internal to a proof* if all the signed formulae of the rule are all

within the proof and its subproofs. A rule application is external if some signed formula is outside the proof and its subproofs.]

If T→I or F→E applies (internally) to the top proof (or its contraposed form), there is an appropriate index translation which replaces $\{1\}$ by the whole index set b, $\{2\}$ by $\{max(b)+1\}$, and $\{1,2\}$ by $b \cup \{max(b)+1\}$. Further $\{3\}$ is replaced by $\{max(b)+2\}$, $\{2,3\}$ by $\{max(b)+1, max(b)+2\}$, and $\{1,2,3\}$ by $b \cup max(b)+1, max(b)+2\}$, etc.

If T→E(ii), T→E(iii), F→I(ii) or F→I(iii) applies internally to the top proof the same index translation as for T→I or F→E still works.

If T→E(i) applies (internally or externally) to an entailment of form T$C \to D_\emptyset$ there is a simple repetition of index for C and D.

(β) For the case where b is $\emptyset$, we replace $\{1\}$ by $\emptyset$ and put $max(b)$ as 0 for the index replacements. I.e., $\{2\}$ is replaced by $\{1\}$, $\{1,2\}$ by $\{1\}$, $\{3\}$ by $\{2\}$, $\{2,3\}$ by $\{1,2\}$, and $\{1,2,3\}$ by $\{1,2\}$, etc.

(II) $a = \{j,\ldots,k\}$, $k \geq 2$, $1 \leq j \leq k-1$, $a - \{k\} = \{j\ldots,k-1\}$, $b = \{k\}$ and $c = \{j,\ldots,k\} = a$, with T→E(ii) applied. No index adjustment is needed, with T→E(ii) being applied *en bloc*.

(III) $a = \{j,\ldots,k\}$, $k \geq 2$, $1 \leq j \leq k-1$, $a - \{k\} = \{j\ldots,k-1\}$, $b = c = \{j,\ldots,k\} = a$. Since $b = c$ is non-null, we just replace each index $\{k\}$ in the top proof by $\{j,\ldots,k\}$ and the *en bloc* application of T→E(ii) by T→E(iii), maintaining the index set. Any occurrence of the index set $\{k\}$ in any subproofs of the top proof would also have to be replaced by $\{j,\ldots,k\}$.

F → .

$$
\begin{array}{l|l}
 & \mathrm{F}C \to D_{a-\{k\}}. \\
 & \quad . \\
 & \quad . \\
 & \quad \mathrm{T}A_b \text{ (or } \mathrm{F}B_b) \\
 & \quad \mathrm{T}B_c \text{ (or } \mathrm{F}A_c) \qquad [\mathrm{F}{\to}\mathrm{I} \text{ for } \mathrm{F}A{\to}B] \\
 & \quad . \\
 & \quad . \\
 & \mathrm{F}A \to B; \ldots_{a-\{k\}} \qquad [\mathrm{F}{\to}\mathrm{E} \text{ for } \mathrm{F}C{\to}D] \\
 & \quad \mathrm{T}A_{\{k\}} \\
 & \quad . \\
 & \quad . \qquad [\mathrm{F}{\to}\mathrm{I} \text{ for } \mathrm{F}E{\to}G] \\
 & \quad \mathrm{T}B_a \\
 & \mathrm{F}E \to G; , \ldots_{a-\{k\}} \qquad [\mathrm{F}{\to}\mathrm{E} \text{ for } \mathrm{F}A{\to}B]
\end{array}
$$

To eliminate $\mathrm{F}A{\to}B_{a-\{k\}}$, we follow a similar procedure to that for $\mathrm{T}A{\to}B_{a-\{k\}}$, by placing the proof of $\mathrm{T}B_a$ from $\mathrm{T}A_{\{k\}}$ into the space between $\mathrm{T}A_b$ and $\mathrm{T}B_c$ in the strand or thread of proof above, or between $\mathrm{F}B_b$ and $\mathrm{F}A_c$ invoking the Contraposition Lemma. However, unlike for T→, there are only two possible index changes, because F→-formulae do not occur in the main proof.

(II) $a = \{j, \ldots, k\}$, $k \geq 2$, $1 \leq j \leq k-1$, $a - \{k\} = \{j \ldots, k-1\}$, $b = \{k\}$ and $c = \{j, \ldots, k\} = a$, and there is no change in index sets.

(III) $a = \{j, \ldots, k\}$, $k \geq 2$, $1 \leq j \leq k-1$, $a - \{k\} = \{j \ldots, k-1\}$, $b = c = \{j, \ldots, k\} = a$.

As for T→E(iii), we just replace each index $\{k\}$ in the bottom proof by $\{j, \ldots, k\}$ and the *en bloc* application of F→I(ii) by F→I(iii), maintaining the index set. Any occurrence of the index set $\{k\}$ in any subproofs of the bottom proof would also have to be replaced by $\{j, \ldots, k\}$.

Finally, $\mathrm{F}C{\to}D$ is eliminated and $(\mathrm{F}E{\to}G;\ldots),\ldots$ is introduced in the same strand or thread of proof, where $\mathrm{F}E{\to}G;\ldots$, which is in the strand or thread of proof of $\mathrm{F}A{\to}B$, is substituted for $\mathrm{F}A{\to}B$ in $\mathrm{F}A{\to}B;\ldots$.

$$
\sim. \;\begin{array}{|l|l}
\mathrm{F}A_a & \mathrm{T}A_a \\
\mathrm{T}{\sim}A_a \ [\mathrm{T}{\sim}\mathrm{I}] & \mathrm{F}{\sim}A_a \ [\mathrm{F}{\sim}\mathrm{I}] \\
\mathrm{F}A_a \ [\mathrm{T}{\sim}\mathrm{E}] & \mathrm{T}A_a \ [\mathrm{F}{\sim}\mathrm{E}]
\end{array}
$$

Clearly, T$\sim A_a$ and F$\sim A_a$ are eliminable.

&.		
	TA, TB_a	FA_a (or FB_a)
	TA&B_a [T&I]	FA&B_a [F&I]
	TA, TB_a [T&E]^2	FA; FB_a [F&E]

TA&B_a is easily eliminated via either or both of TA_a and TB_a. If only one of TA_a or TB_a is used, the other would be removed from the proof, together with its strand. If both are used, the two strands would continue on. FA&B_a is eliminated either by removing the adjacent thread of proof initiated by FB or by removing the adjacent thread of proof initiated by FA. We do not then need to apply ;E to eliminate the semicolon.

$\vee$.		
	TA_a (or TB_a)	FA, FB_a
	T$A \vee B_a$ [T $\vee$ I]	F$A \vee B_a$ [F $\vee$ I]
	TA; TB_a [T $\vee$ E]	FA, FB_a [F $\vee$ E]^2

The eliminations of T$A \vee B$ and F$A \vee B$ are similar to that of FA&B and TA&B, respectively.

(ii) As indicated above, we also need to consider the →-, &- and ∨-rules when ,I or ;E occur, singly or multiply, in between the introduction and elimination rules of an IE-turning point. As for ;E, replacing ,E in Brady [2006a], we repeat the elimination rules and apply ;E afterwards. Here for ,I, we just repeat the introduction rules, whilst applying ,I beforehand.

By the successive elimination of IE-turning points, as in (i) and (ii) above, we will reach a stage where there are no deviations occurring in the overall proof and thus a normalized proof in NMC is obtained. As such eliminations shrink the length of proofs, this will contribute to the decidability argument given in §4.

R⇒L. Proofs in a normalized NMC are still proofs in MMC, except for uses of the F→-rules.

So, we consider F→I(ii), F→I(iii) and F→E(ii). To replace usage of F→E(ii), F→I(ii) and F→I(iii), we take an F-subproof of TB_a from T$A_{\{k\}}$, including F→I(ii) and F→I(iii) applications from T$C_{\{k\}}$ to TD_a and T$E_{\{k\}}$ to TG_a, etc., the F-subproof being used to eliminate F$A \to B_{a-\{k\}}$. Thus, we introduce F$C {\to} D$, F$E {\to} G, \ldots\ _{a-\{k\}}$, where $a - \{k\} \neq \emptyset$. We then convert the F-subproof to a T-subproof, inserting a proof of T$A {\to}\ B_{a-\{k\}}$ from T$(C{\to}D)\&(E{\to}G)\&\ldots_{a-\{k\}}$. We re-index this proof (if necessary) by putting a as $\{1, 2\}$ and k as 2, so $a - \{k\}$ as $\{1\}$, and replacing the other indices $k + n$ by $2 + n$, for each $n \geq 1$, in its subproofs. By T→I(i), this then yields a proof of T$(C \to D)\&(E \to G)\&\ldots \to .A \to B_\emptyset$, which when applied to the original F$A \to B_{a-\{k\}}$ yields F$(C \to D)\&(E \to G)\&\ldots_{a-\{k\}}$ by T→E(i). Repeated applications of F&E then yield F$C{\to}D$;F$E \to G$;$\ldots_{a-\{k\}}$, as required. This is illustrated below, with the inclusion of F→I(iii) and T→E(iii).

The Replacement of F→E(ii), F→I(ii) and F→I(iii).

$$
\begin{array}{l|l}
 & \mathrm{F}A \to B_{a-\{k\}} \\
 & \quad \mathrm{T}A_{\{k\}} \\
 & \quad \vdots \\
 & \quad \mathrm{T}C, \mathrm{T}E_{\{k\}} \\
 & \quad \mathrm{T}D, \mathrm{T}G_a \qquad [\mathrm{F}{\to}\mathrm{I(ii)}] \\
 & \quad \vdots \\
 & \quad \mathrm{T}H_a \\
 & \quad \mathrm{T}J_a \qquad [\mathrm{F}{\to}\mathrm{I(iii)}] \\
 & \quad \vdots \\
 & \quad \mathrm{T}B_a \\
 & \mathrm{F}C \to D; \mathrm{F}E \to G; \mathrm{F}H \to J_{a-\{k\}} \qquad [\mathrm{F}{\to}\mathrm{E(ii)}]
\end{array}
$$

$$
\begin{array}{ll}
\quad \mathrm{T}(C \to D)\&(E \to G)\&(H \to J)_{\{1\}} & \\
\quad\quad \mathrm{T}A_{\{2\}} & \\
\quad\quad \vdots & \\
\quad\quad \mathrm{T}C, \mathrm{T}E_{\{2\}} & \\
\quad\quad \mathrm{T}D, \mathrm{T}G_{\{1,2\}} & [\mathrm{T}{\to}\mathrm{E(ii)}] \\
\quad\quad . & \\
\quad\quad \mathrm{T}H_{\{1,2\}} & \\
\quad\quad \mathrm{T}J_{\{1,2\}} & [\mathrm{T}{\to}\mathrm{E(iii)}] \\
\quad\quad . & \\
\quad\quad \mathrm{T}B_{\{1,2\}} & \\
\quad \mathrm{T}A \to B_{\{1\}} & [\mathrm{T}{\to}\mathrm{I(ii)}] \\
\mathrm{T}(C \to D)\&(E \to G)\&(H \to J) \to .A \to B_{\emptyset} & [\mathrm{T}{\to}\mathrm{I(i)}] \\
\quad \mathrm{F}A \to B_{a-\{k\}} & \\
\quad \mathrm{F}(C \to D)\&(E \to G)\&(H \to J)_{a-\{k\}} & [\mathrm{T}{\to}\mathrm{E(i)}] \\
\quad \mathrm{F}C \to D; \mathrm{F}E \to G; \mathrm{F}H \to J_{a-\{k\}} & [\mathrm{F\&E}]
\end{array}
$$

The applications of F→E(ii), F→I(ii) and F→I(iii) above are replaced by the corresponding derivation of $\mathrm{T}(C \to D)\&(E \to G)\&(H \to J) \to .A \to B_{\emptyset}$ with the re-indexing, using T→I rules, following this with the application of the T→E(i) and FE rules.

Note that the resultant proof in MMC is not a normal proof in that it features an application of T→I(i), followed by an application of T→E(i) to its conclusion. However, the F-subproof is simply renamed as a T-subproof with the addition of a few standard steps, and so this would not affect the decidability result to follow, allowing MMC to be used for deciding theoremhood along with that of NMC.

4 Properties of the normalization of NMC.

We first prove the subformula property for NMC, which will hold the key to decidability.

Theorem 4. (Subformula Theorem)
For any theorem $\mathrm{T}A_{\emptyset}$ of a normalized NMC, any formula instance B occurring as $\mathrm{T}B_a$ or $\mathrm{F}B_a$ in this proof is a subformula of A.

Proof. Given such a proof of $TA_\emptyset$, we use induction on the depth k of the subproof containing the formula instance B.

Depth = 0. By the N.P.P. Lemma, a normalized main proof has introduction rules only, starting with T→-formulae, introduced by application of T→I(i). Hence, each formula in the main proof will be a subformula of all later formulae in the proof, thus including the final theorem A.

Depth $= k + 1$. As an induction assumption, we let all formula instances in subproofs of depth k be subformulae of the final theorem A. This will include all T→- and F→-formulae, which are the only formulae that can interact with subproofs S of depth $k + 1$. Indeed, the hypothesis and conclusion of such a subproof S are immediate subformulae of one of these T→- and F→-formulae of depth k, due to the application of a T→I rule. Further, any application of T→E or F→I into S will be made up of immediate subformulae of T→- or F→-formulae from depth k. By the N.P.P. Lemma, the rules of S applied in between the hypothesis of S and a premise of a T→E or F→I rule or in between a conclusion of a T→E or F→I rule and a premise of a further T→E or F→I rule, or in between the conclusion of a T→E or F→I rule and the conclusion of S will be a sequence of connective elimination rules, followed by a sequence of connective introduction rules, with possible use of the rules ,I and ;E, included to introduce strands of proof or to close threads of proof. So, for each application of an elimination or introduction rule, the formula reached will be a subformula of a formula in the preceding subproof of depth k, and hence a subformula of the final theorem A. In the case where a T→- or F→-formula(e) is reached in S, as the initial strand(s) of a T→-cluster, followed by an application of T→E or F→I into a subproof of depth $k+2$, the T→- or F→-formula(e) will be preceded by an elimination rule involving a subformula(e) of A. Also, in the case where a T→- or F→-formula is reached in S, as the latter part of a T→-cluster, preceded by an application of T→I or F→E from a subproof of depth $k+2$, the T→- or F→-formula will be followed by an introduction rule involving a subformula of A.

Thus, by induction on the depth of subproofs, we have shown that each formula instance in the proof of the final theorem A is indeed a subformula of A.

Before moving on, we prove a corollary limiting the placement of the subformulae of a final theorem within its proof. We first define the degree of a formula C, $deg(C)$, inductively as follows:

(i) $deg(p) = 0$, for all propositional variables p.

(ii) If $deg(C) = m$ then $deg({\sim}C) = m$.

(iii) If $deg(C) = m$ and $deg(D) = n$ then $deg(C\&D) = deg(C \vee D) = max\{m, n\}$.

(iv) If $deg(C) = m$ and $deg(D) = n$ then $deg(C \to D) = max\{m, n\} + 1$.

We next inductively define the depth of a subformula occurrence C in a formula A, $d(C, A)$, by starting with A and by considering each immediate subformula or pair of immediate subformulae in turn, as follows:

(i) $d(A, A) = 0$.

(ii) If $d({\sim}C, A) = n$ then $d(C, A) = n$.

(iii) If $d(C\&D, A) = n$ then $d(C, A) = d(D, A) = n$.

(iv) If $d(C \vee D, A) = n$ then $d(C, A) = d(D, A) = n$.

(v) If $d(C \to D, A) = n$ then $d(C, A) = d(D, A) = n + 1$.

It is easy to show that the maximum depth of a subformula occurrence in a formula A is the degree of A, i.e. $max\{d(C, A) : C \text{ is a subformula occurrence in } A\} = deg(A)$.

Corollary. If C is a subformula instance occurring in a structure of index a in a normalized proof of a final theorem $\mathrm{T}A_\emptyset$ in NMC then this occurrence of C lies in a subproof whose depth, $max(a)$, is equal to $d(C, A)$. Here, we take $max(\emptyset)$ to be 0, as the depth of the main proof is 0. [This aligns the two concepts of 'depth'.]

Proof. It is already clear that in a subproof, with hypothesis $\mathrm{T}C_{\{k\}}$, the only index sets that can be generated by the rules of NMC are $\{k\}$ and $\{j, \ldots, k\}$ (with $1 \le j \le k - 1$), both with a maximum of k, the depth of the subproof. Also, in the main proof, the only index set is $\emptyset$, and $max(\emptyset) = 0$. So, it remains to show that the depth of any proof containing C is equal to $d(C, A)$. We proceed by induction on the depth of proofs.
(i) Let C occur in the main proof. Since only ∼-, &- and ∨-rules are used within the proof, together with the depth of any such formula C in A remains at 0.
(ii) For all formulae D occurring in proofs of depth k, let $k = d(D, A)$. We let C occur in a subproof S of depth $k + 1$. The hypothesis and conclusion of S are of depth $k + 1$ in A, since their generating entailment formula (through T→I or F→E) is of depth k in A. The application of ∼-, &- and ∨-rules in S do not change the depth of their formulae in A, and neither do ,I nor ;E. Also, the T→E and F→I rules, when applied into S, apply to formulae of depth $k + 1$ in A, as the entailments of these rules are of depth k. Further, the components of →-formulae in S do not occur in S, but may occur in a further subproof. So, any formula C, occurring in S, must have a depth of $k + 1$ in A.

Thus, by induction on the depth of proofs, all the formulae occurring in a proof of depth k have a subformula depth of k in A.

Theorem 5. (Decidability Theorem)
MC is decidable.

Proof. In order to prove decidability, we need to ensure there is a finite limit on the number of overall (finite) proof attempts for a final formula A in the main proof. Such a finite limit can be determined by restricting such proof attempts by using key properties of the proofs of theorems in NMC. The previous two results are critical. Theorem 4 shows that any formulae that can appear in a proof of a final theorem A are restricted to those that are subformulae of A. The Corollary to Theorem 4 shows us that any such subformula can only appear at a depth of subproof equal to its depth in A. So, if the degree of A is d then d is the maximum depth of subformula occurrences in A and hence subproofs are limited to those with a depth less than or equal to d. This also limits the usable index sets to complete subsets of $\{1, \ldots, d\}$.

We define the length of a structure in a subproof S as the number of signed formulae in the structure. Due to the N.P.P. Lemma, this length is limited by one plus the number of applications of **either** the connective rules, T∨E **or** F&E for the elimination phases applied to subformula occurrences either of the hypothesis of the subproof containing the structure or of the consequents of T→E or F→I rules applied to their immediate superproof, **or** subsequent connective rules T&I or F∨I for the introduction phases applied to subformula occurrences **either** of the antecedents of T→E or F→I rules applied to their immediate superproof or of the conclusion of the subproof containing the structure. So, the length of a structure in a subproof of depth k is limited by one plus the number of conjunctions and disjunctions of depth k in the final theorem A. Further, the number of structures in a subproof S of depth k is limited by the number of possible signed formulae of depth k in A, and determined by the T-clusters occurring in its immediate superproof, which indeed supply the starting and finishing structures of all the n.p.p.'s of the subproof S. Note here that repetition of a structure in a subproof induces an otiose piece of proof. Further, the number of possible subproofs of depth k, which are sequences of indexed structures with index sets of form: $\{k\}$ and $\{j, \ldots, k\}$, is limited. Note too that the repeating of an entire subproof of depth k in an overall proof attempt is otiose. Hence, the number of overall proof attempts for the final theorem A is limited and the logic MC is decidable.

This decidability result is new as it differs from §12.6 of [2003], essentially repeated in [2003a], where the Routley-Meyer semantics is used to show the decidability of MC plus the distribution axiom, viz. DJ^{d}. (Indeed, such a distribution axiom cannot be removed from the Routley-Meyer semantics. See Brady [2022] on this.) Further, the use of Gentzen systems to prove decidability is problematic due to the Gentzenized form of the Conjunctive Syllogism axiom A10 using fusion , viz. $A \circ B \to A \circ (A \circ B)$, being a form of contraction, where $\circ$ satisfies the two-way rule, $A \circ B \to C \Leftrightarrow A \to .B \to C$. This can be better seen in its deductive equivalent, $A \circ (A \circ B) \to C \Rightarrow A \circ B \to C$, with contraction occurring on the antecedent side.

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