From depth relevance to connexivity

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Abstract

Brady [4] used the matrix for Meyer's crystal lattice **CL** to build a hierarchical model structure for his deep relevant logic $\mathbf{DR^d}$. In this paper I modify the matrix for **CL** so as to define a connexive conditional. In doing so, I arrive at a family of connexive logics satisfying the depth relevance property. This results in a way to satisfactorily combine connexivity and relevance without trivializing the logic and without validating unappealing theorems.

Keywords: Relevant logics; connexive logics; sociative logics; depth relevance; naïve set theory; inconsistent mathematics.

1 Introduction

Ross Brady devoted decades of his life to developing a logic which could support naïve set theory. Naïve set theory [62] is characterized by the Unrestricted Comprehension axiom that stems from the foundational works of Frege, Cantor and Dedekind. This axiom guarantees that there exists a set whose elements are exactly those which satisfy a given property, for whatever property expressible in the language —which lead to disaster in the case of Frege, who used classical logic as a basis of his set theory. Unlike Frege, Brady's efforts focused on relevant logics of the *strongly paraconsistent* kind, i.e. those which, besides being able to tolerate contradictions without trivializing the theory, were able to work with the Unrestricted Comprehension axiom $\exists y \forall x(x \in y \leftrightarrow \varphi x)$, for arbitrary properties φ, without falling prey to Curry's paradox [48]. The latter is an argument with many variants showing that, from the Unrestricted Comprehension axiom and well-known principles for $\rightarrow$, like Contraction, $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$, one can derive an arbitrary formula A, thus trivializing any naïve set theory with the logical principles involved in such reasoning [45, pp.289-90].

Relevant logics like **T**, **R** or **E** validate Contraction and the other logical principles involved in Curry's paradox. So none of these well-known systems

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could be a non-trivial foundation for naïve set theory. This lead Brady to explore *deep relevant* logics, i.e. those that guarantee that any implicative formula which is a theorem is such that the antecedent shares a variable with the consequent at the same depth. This property is stronger than that which **T**, **R** or **E** satisfy, to wit, that the antecedent and consequent of an implicational theorem share at least one propositional variable, regardless of depth. The concept of depth of a subformula in a formula was key to separate such relevant systems in a non *ad hoc* way so that one could invalidate crucial principles in Curry's paradox, like Contraction. Thus, Brady developed his universal logic $\mathbf{DJ^dQ}$, both deep relevant and strongly paraconsistent, and built a non-trivial naïve set theory on it [9].[1]

Depth relevance is an excellent means to obtain relevant logics. In fact, the notion of relevance between antecedents and consequents is a particular case of the sort of relations one may impose on sound implications. The family of *sociative logics* [50], which includes, for instance, relevant logics and connexive logics, consists of those systems whose implication connective requires that antecedent and consequent are associated through a special kind of relation which bears on their implicational grounds. While the relevance relation is a positive sort of relation demanding, at least, that all propositions in an antecedent are used to derive the consequent, other relations one can impose on conditionals can be negative, like the one we find in connexive logics. It is known that Chrysippus's account of a sound implication, championed by McCall as the standard view of connexive implication [28, p.8-12], demands that the negation of the consequent is incompatible with the antecedent. Hence, one can naturally expect an overlap between connexive and relevant implication by imposing both sorts of relations to valid implicative formulas: relevance between antecedent and consequent, and incompatibility between antecedent and the negation of the consequent. However, combining relevance and connexivity is not a straightforward task, since either triviality, inconsistency or unreasonable theorems may arise in attempting so [34], [45, pp.245-6], [54].

But depth relevance, as it turns out, is not exclusive to the weaker systems of relevant logics, ranging from the basic system **B** to Brady's $\mathbf{DR^d}$ from [4]. Méndez and Robles' work on weak relevant matrices [32] and weak relevant model structures [41] shows that there are logics satisfying the depth relevance property which neither include nor are included in **RM** —an important extension of **R**, which sits in the stronger side of the spectrum of relevant logics and contains the weaker systems mentioned above. Thus, depth relevance is a more general property than originally thought. As a consequence of these results, one can have deep relevant systems that are also connexive; moreover, one can obtain systems of this sort which avoid triviality and avoid validating unreasonable theorems, thus providing a class of sociative logics that satisfactorily combine relevance and connexivity.

[1]For proofs on the non-triviality of naïve set theory based on a stronger logic **DSQ** which contains $\mathbf{DJ^dQ}$, see [7]; see also [3] and [10].

In this paper I study how to obtain logical systems with the depth relevance property that also validate the characteristic connexive theses. Indeed, such theses respect the depth relevance property, so it is natural to ask if they can be added to logics like $\mathbf{DR^d}$ with as few modifications as possible so as to get a non-trivial logic which would also result in a strongly paraconsistent logic suitable for a foundation of naïve set theory. In Section 2, I introduce the necessary background and the model structures helping one to establish depth relevance; and in Section 3, I provide a slight modification to such model structures that allows for connexive theses to hold in the background logic. In the conclusions section, I briefly discuss the prospects of connexive naïve set theory, which I will thoroughly analyze in a future paper.

2 Depth relevance and wr-model structures

In [4], Brady presented the logic $\mathbf{DR^d}$, whose axioms and rules of derivation are as follows —where $\sim$, $\wedge$ and $\rightarrow$ are primitive connectives, $p, q, r, \ldots$ are sentential variables, $A, B, C, \ldots$ are arbitrary formulas, $A \vee B \stackrel{df}{=} \sim(\sim A \wedge \sim B)$ and $A \leftrightarrow B \stackrel{df}{=} (A \rightarrow B) \wedge (B \rightarrow A)$:

A1. $\vdash A \rightarrow A$
A2. $\vdash (A \wedge B) \rightarrow A$
A3. $\vdash (A \wedge B) \rightarrow B$
A4. $\vdash ((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$
A5. $\vdash ((A \rightarrow B) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C)$
A6. $\vdash (A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$
A7. $\vdash \sim\sim A \rightarrow A$
A8. $\vdash (A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)$
A9. $\vdash A \vee \sim A$
R1. $\vdash A, \vdash A \rightarrow B \therefore \vdash B$
R2. $\vdash A, \vdash B \therefore \vdash A \wedge B$
R3. $\vdash C \vee A, \vdash C \vee (A \rightarrow B) \therefore \vdash C \vee B$
R4. $\vdash C \vee A \therefore \vdash C \vee \sim(A \rightarrow \sim A)$
R5. $\vdash E \vee (A \rightarrow B), \vdash E \vee (C \rightarrow D) \therefore \vdash E \vee ((B \rightarrow C) \rightarrow (A \rightarrow D))$

Rules R3.–R5. are called *disjunctive rules* and the superscript $\mathbf{d}$ in $\mathbf{DR^d}$ indicates that (some of) these are included in the logic; so $\mathbf{DR}$ is just $\mathbf{DR^d}$ without disjunctive rules. This convention will be used throughout the paper for other systems.

Brady's main goal was to introduce a family of relevant logics which he calls *strongly paraconsistent* —i.e. which reject the inference $\vdash A, \vdash\sim A \therefore \vdash B$ and which, moreover, reject the Contraction law $\vdash (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$. Such family of logics provides suitable logical bases for naïve set theory [62] because, on the one hand, the Unrestricted Comprehension axiom guarantees the existence of contradictory sets (like Russell's), and, on the other hand, rejecting

Contraction is key to dissolve Curry's paradox [48] —which trivializes even simply paraconsistent naïve set theories. Systems on the vecinity of **DK** and **DJ** are Brady's preferred strongly paraconsistent logics for this endeavour —see [9].

It is traditionally posited that a necessary condition for relevant logics is the Variable Sharing Property (VSP).

Definition 2.1 (Variable-sharing property (VSP))**.** A logic **L** has the *variable-sharing property* (VSP) iff, if $A \to B$ is a theorem of **L**, then A and B share at least one propositional variable.

However, relevant logics like **E**, **R** and **T** have the VSP but validate Contraction and other schemas like Pseudo-Modus Ponens, $\vdash (A \wedge (A \to B)) \to B$, which allow Curry's paradox to be derived in naïve set theory. A stronger property is needed to keep variable sharing but reject Contraction and Pseudo-Modus Ponens. This is Brady's *depth relevance* property.

Definition 2.2 (Depth relevance property (DRP))**.** A logic **L** has the *depth relevance property* (DRP) iff, if $A \to B$ is a theorem of **L**, then A and B share at least one propositional variable at the same depth.

Roughly, the depth of an occurrence of a subformula B in a formula A is the number of nested conditionals required to reach the occurrence of B in A. We can make this (and the related notion of the degree of a formula) precise via the following inductive definitions.

Remark 2.1. I follow Robles and Méndez in the notational practice of distinguishing conditionals that are sensitive to depth of subformulas in a formula against those which are not. I write $\twoheadrightarrow$ for depth-sensitive conditionals and $\to$ for those which do not take depth into account.

Definition 2.3 (Degree of formulas)**.** Let $\deg(A)$ be read as "the degree of formula A" and denote a natural number $n \geq 0$. We define deg inductively:

1. If A is a propositional variable, then $\deg(A) = 0$
2. If A is of the form $\sim B$ and $\deg(B) = n$, then $\deg(A) = n$
3. If A is of the form $B \vee C$ (or $B \wedge C$ or $B \to C$) and $\deg(B) = m$ and $\deg(C) = n$, then $\deg(A) = \max\{m, n\}$
4. If A is of the form $B \twoheadrightarrow C$ and $\deg(B) = m$ and $\deg(C) = n$, then $\deg(A) = \max\{m, n\} + 1$

Definition 2.4 (Depth of a subformula within a formula)**.** Let $d[X, Y]$ be read as "the depth of the subformula X within the formula Y" and denote a natural number $n \geq 0$. We define d inductively:

1. $d[A, A] = 0$
2. If $d[\sim B, A] = n$, then $d[B, A] = n$
3. If $d[B \wedge C, A] = n$ (or $d[B \vee C, A] = n$ or $d[B \to C, A] = n$), then $d[B, A] = d[C, A] = n$

4. If $d[B \twoheadrightarrow C, A] = n$, then $d[B, A] = d[C, A] = n + 1$

Example 2.1. Let X be the formula $p \twoheadrightarrow ((p \twoheadrightarrow q) \twoheadrightarrow q)$, an instance of Assertion, $A \twoheadrightarrow ((A \twoheadrightarrow B) \twoheadrightarrow B)$. Then $\deg(p) = 0 = \deg(q)$ so $\deg(p \twoheadrightarrow q) = 1$, $\deg((p \twoheadrightarrow q) \twoheadrightarrow q) = 2$ and $\deg(X) = 3$. Let us enumerate each occurrence of p and q in X: $p_1 \twoheadrightarrow ((p_2 \twoheadrightarrow q_1) \twoheadrightarrow q_2)$. We have, $d[X, X] = 0$ so $d[p_1, X] = 1 = d[(p_2 \twoheadrightarrow q_1) \twoheadrightarrow q_2, X]$ and then $d[p_2 \twoheadrightarrow q_1, X] = 2 = d[q_2, X]$ so $d[p_2, X] = 3 = d[q_1, X]$. We see that the antecedent of X shares one variable with its consequent, to wit, p; however, p has depth 1 in the antecedent and depth 3 in the consequent. So while X, or, more generally, Assertion, is valid in relevant systems like **R**, it cannot be valid in systems with the DRP, like $\mathbf{DR^d}$.

In fact, the DRP implies two strong properties from relevant systems, to wit:

Definition 2.5 (Ackermann property (AP))**.** A logic **L** has the *Ackermann property* (AP) if, for any A, B, C, the formula $A \to (B \to C)$ is unprovable in **L** if A does not contain an implicative formula.

Definition 2.6 (Converse Ackermann property (CAP))**.** A logic **L** has the *converse Ackermann property* (CAP) if, for any A, B, C, the formula $(A \to B) \to C$ is unprovable in **L** if C does not contain an implicative formula.

Theorem 2.1. Let **L** be a logic with the DRP. Then, **L** has the AP and the CAP. [41, p.111]

The original semantics for $\mathbf{DR^d}$ depends on the logical matrix for Meyer's crystal lattice **CL** [56, p.98]. We first give some useful definitions for logical matrices, following [32].

Definition 2.7 (Logical matrix)**.** A *logical matrix* $\mathtt{M}$ is a structure

$$(K, T, F, f_\sim, f_\wedge, f_\vee, f_\to)$$

where: (1) K is a set; (2) T and F are non-empty subsets of K such that $T \cup F = K$ and $T \cap F = \emptyset$; and (3) $f_\wedge, f_\vee, f_\to$ are binary functions (distinct of each other) on K, and $f_\sim$ is a unary function on K.

Intuitively, K is a set of truth values, T is the set of designated values, F is the set of non-designated values, and the f's are truth functions defined on K^n interpreting the connective appearing in them as a subindex, where $n \geq 0$ is the arity of the connective. Note that it is not necessary for a matrix to have all or exactly those connectives represented by the f's above for it to be a logical matrix.

Definition 2.8. Let $\mathtt{M}$ be a logical matrix. $\mathtt{M}$ *verifies* a formula A iff for any assignment, v_m, of elements of K to the propositional variables of A, $v_m(A) \in T$. $\mathtt{M}$ *falsifies* A iff $\mathtt{M}$ does not verify A.

Definition 2.9. Let $\vdash A_1, \ldots, \vdash A_n \therefore \vdash B$ be a rule of derivation, and let `M` be a logical matrix. Then, `M` *verifies* $\vdash A_1, \ldots, \vdash A_n \therefore \vdash B$ iff for any assignment, v_m, of elements of K to the variables of $A_1, \ldots, A_n$ and B, if $v_m(A_1) \in T$, …, $v_m(A_n) \in T$, then $v_m(B) \in T$. `M` *falsifies* $\vdash A_1, \ldots, \vdash A_n \therefore \vdash B$ iff `M` does not verify $\vdash A_1, \ldots, \vdash A_n \therefore \vdash B$.

Definition 2.10. Let `M` be a logical matrix and **L** be a logic. `M` *verifies* **L** iff `M` verifies all axioms and rules of derivation of **L**.

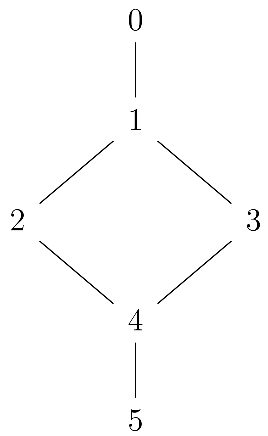
The logical matrix $\mathtt{M}^{\mathbf{CL}}$ for Meyer's crystal lattice **CL** [56, p.98] has $K = \{0, 1, 2, 3, 4, 5\}$, $T = \{0, 1, 2, 3, 4\}$, $F = \{5\}$, and the truth functions for its connectives are defined through the following tables.

$\to$	0	1	2	3	4	5	$\sim$
*0	0	5	5	5	5	5	5
*1	0	4	5	5	5	5	4
*2	0	2	2	5	5	5	2
*3	0	3	5	3	5	5	3
*4	0	1	2	3	4	5	1
5	0	0	0	0	0	0	0

$\wedge$	0	1	2	3	4	5
*0	0	1	2	3	4	5
*1	1	1	2	3	4	5
*2	2	2	2	4	4	5
*3	3	3	4	3	4	5
*4	4	4	4	4	4	5
5	5	5	5	5	5	5

$\vee$	0	1	2	3	4	5
*0	0	0	0	0	0	0
*1	0	1	1	1	1	1
*2	0	1	2	1	2	2
*3	0	1	1	3	3	3
*4	0	1	2	3	4	4
5	0	1	2	3	4	5

The Hasse diagram below explains the name of this algebraic structure.



One neat property of $\mathtt{M}^{\mathbf{CL}}$ is that it validates the theorems of **R** and can be used to show that a logic has the VSP. In [41, p.114], this matrix has been shown to be a special case of the more general notion of *weak relevant matrix*.

Definition 2.11 (wr-matrix)**.** Let $\mathtt{M}$ be a logical matrix in which x_i, x_r and x_F are elements of K distinct of each other such that $x_F \in F$ and the following conditions are fulfilled:

1. $f_\wedge(x_i, x_i) = f_\vee(x_i, x_i) = f_\to(x_i, x_i) = f_\sim(x_i) = x_i$
2. $f_\wedge(x_r, x_r) = f_\vee(x_r, x_r) = f_\to(x_r, x_r) = f_\sim(x_r) = x_r$
3. $f_\to(x_i, x_r) = x_F$

It is said that $\mathtt{M}$ is a *weak relevant* matrix (wr-matrix, for short).

Proposition 2.1. $\mathtt{M}^{\mathbf{CL}}$ is a wr-matrix.

Proof. We take $x_i = 2$, $x_r = 3$ and $x_F = 5$; these satisfy the conditions for a wr-matrix. □

Since $\mathtt{M}^{\mathbf{CL}}$ verifies **R**, from the previous proposition and the following theorem we know that **R** has the VSP.

Theorem 2.2. Let $\mathtt{M}$ be a wr-matrix and let **L** be a logic verified by $\mathtt{M}$. Then **L** has the VSP. [41, p.112]

$\mathbf{DR^d}$ uses a *hierarchical model structure* defined as an indexed set of **CL**-matrices. This allows to filter the theorems of **R** in a way that we only keep those that satisfy the DRP. Again, since $\mathtt{M}^{\mathbf{CL}}$ is a wr-matrix, we may follow [41, p.114] in defining the general corresponding notion of *wr-model structure.*

Definition 2.12 (wr-model structure)**.** Let $\mathtt{M}$ be a wr-matrix. A *wr-model structure* $[\![\mathtt{M}]\!]$ is the set $\{\mathtt{M}_0, \mathtt{M}_1, \mathtt{M}_2, \ldots, \mathtt{M}_n, \ldots \mathtt{M}_\omega\}$, where $\mathtt{M}_0, \mathtt{M}_1, \mathtt{M}_2, \ldots, \mathtt{M}_n, \ldots \mathtt{M}_\omega$ are all identical matrices to the wr-matrix $\mathtt{M}$.

The idea is to have a hierarchy of levels, all of which match possible depths of subformulas in a formula. At each level, a corresponding valuation and interpretation of propositional variables and formulas can be given, so as to evaluate subformulas taking their depth into account. This is made precise as follows.

Definition 2.13 (Valuations and interpretations in a wr-model structure)**.** A *valuation* v in a wr-model structure $[\![\mathtt{M}]\!]$ consists of a valuation v_j for all propositional variables, for each wr-matrix $\mathtt{M}_j$ $(0 \leq j \leq \omega)$. Each v_j assigns one of the elements of K to each propositional variable. Then, each valuation v is extended to an interpretation I consisting of the interpretations I_j for all atomic formulas, for all j $(0 \leq j \leq \omega)$, which are given as follows: for all propositional variables p and formulas A, B,

1. $I_j(p) = v_j(p)$
2. $I_j(\sim A) = \sim(I_j(A))$
3. $I_j(A \wedge B) = I_j(A) \wedge I_j(B)$
4. $I_j(A \vee B) = I_j(A) \vee I_j(B)$
5. $I_j(A \to B) = I_j(A) \to I_j(B)$

where 1.–5. are evaluated according to the wr-matrix $\mathtt{M}$. In addition, formulas of the form $A \twoheadrightarrow B$ are evaluated as follows:

6.1. for $j = 0$, $I_0(A \twoheadrightarrow B) = x_T$, where x_T is some designated value we fix for all interpretations
6.2. for $0 < j < \omega$, $I_j(A \twoheadrightarrow B) = I_{j-1}(A \rightarrow B)$
6.3. for $j = \omega$, $I_\omega(A \twoheadrightarrow B) \in T$ iff $I_j(A \rightarrow B) \in T$ for all j $(0 \leq j \leq \omega)$.

Clauses 1.–5. for interpretations in a wr-model only ask for us to use the valuations of the propositional variables at the corresponding level and the truth tables from $\mathtt{M}$, so no difference, particularly regarding $\rightarrow$, will result between simply using $\mathtt{M}$ in the regular way as oppossed to using the wr-model structure. The difference arises when we take 6.1.–6.3. into account, whose effect is that of turning $\rightarrow$ into a depth-sensitive connective $\twoheadrightarrow$ that can now be evaluated together with other valuations and interpretations of a different depth level. This is what allows $\mathbf{DR^d}$ to filter those $\mathbf{R}$-theorems that do not satisfy the DRP.

Finally, validity in a wr-model structure is defined as follows.

Definition 2.14 (Validity in a wr-model structure)**.** Let $[\![\mathtt{M}]\!]$ be a wr-model structure, let $B_1, \ldots, B_n, A$ be formulas and let $\mathbf{L}$ be a logic. A is *valid* in $[\![\mathtt{M}]\!]$, i.e. $\models_{[\![\mathtt{M}]\!]} A$, iff $I_\omega(A) \in T$ for all valuations v. The rule $\vdash B_1, \ldots, \vdash B_n \therefore \vdash A$ *preserves* $[\![\mathtt{M}]\!]$-validity iff, if $I_\omega(B_1) \in T, \ldots, I_\omega(B_n) \in T$, then $I_\omega(A) \in T$ for all valuations v. Finally, $[\![\mathtt{M}]\!]$ *verifies* $\mathbf{L}$ iff all axioms of $\mathbf{L}$ are $[\![\mathtt{M}]\!]$-valid and all rules of $\mathbf{L}$ preserve $[\![\mathtt{M}]\!]$-validity.

Using the above definitions and notation, we say that Brady's $\mathbf{DR^d}$ is verified precisely by the wr-model structure $[\![\mathtt{M}^{\mathbf{CL}}]\!] = \{\mathtt{M}^{\mathbf{CL}}_0, \mathtt{M}^{\mathbf{CL}}_1, \mathtt{M}^{\mathbf{CL}}_2, \ldots, \mathtt{M}^{\mathbf{CL}}_n, \ldots \mathtt{M}^{\mathbf{CL}}_\omega\}$. Thus, though Contraction and Pseudo-Modus Ponens always receive designated values using $\mathtt{M}^{\mathbf{CL}}$, they do not if we check for validity in $[\![\mathtt{M}^{\mathbf{CL}}]\!]$. In fact, Robles and Méndez generalize Brady's Lemma 2 from [4] to get such counterexamples as follows.

Lemma 2.1. Let $[\![\mathtt{M}]\!]$ be a wr-model structure and let $A \twoheadrightarrow B$ be a formula such that A and B do not share a propositional variable at the same depth. Then there is some interpretation I_k in $[\![\mathtt{M}]\!]$ such that for each subformula C of A, $I_k(C) = x_i$ and, for each subformula D of B, $I_k(D) = x_r$.

Proof. The result is a particular case of the more general Lemma 5.6 from [41, p.115]. □

Example 2.2. Let X be the formula $(p \wedge (p \twoheadrightarrow q)) \twoheadrightarrow q$, an instance of Pseudo-Modus Ponens. Let A be $p \wedge (p \twoheadrightarrow q)$, the antecedent of X; and let B be q, the consequent of X. We have $\deg(X) = 2$, $\deg(A) = 1$, $\deg(B) = 0$, $d[p_1, A] = 0$, $d[p_2, A] = 1 = d[q, A]$ and $d[q, B] = 0$ —where p_1 and p_2 are, respectively, the first and the second occurrence of p in A. We use $[\![\mathtt{M}^{\mathbf{CL}}]\!]$ to provide a valuation v_{m-d-1}, where $m = \deg(X)$ and d is the depth of each propositional variable in A or in B. Since Lemma 2.1 is always witnessed by $k = \deg(A \twoheadrightarrow B) - 1$, this valuation can be extended to an interpretation I_{m-1} that assigns 2 to A and 3 to B, so that the value of X is 5, thus:

1. $v_{2-1-1}(p) = 2 = v_{2-1-1}(q)$
2. $v_{2-0-1}(p) = 2$ and $v_{2-0-1}(q) = 3$

Then $I_0(p \to q) = 2$ so $I_1(p \twoheadrightarrow q) = 2$, and then $I_1(p \wedge (p \twoheadrightarrow q)) = 2$, but we also have $I_1(q) = 3$ so $I_1((p \wedge (p \twoheadrightarrow q)) \to q) = 5$, whence $I_2((p \wedge (p \twoheadrightarrow q)) \twoheadrightarrow q) = 5$. So Pseudo-Modus Ponens is invalid in $\mathbf{DR^d}$.

In turn, Lemma 2.1 allows Robles and Méndez to generalize Brady's Theorem 1 from [4] as the following theorem which will come in handy in the next section. Since Lemma 2.1 gives counterexamples to implicational formulas that do not share a variable between antecedent and consequent at the same depth, by contraposing then one implicational formula with no counterexamples will be such that antecedent and consequent share a variable at the same depth.

Theorem 2.3. Let $[\![\mathtt{M}]\!]$ be a wr-model structure and suppose $\models_{[\![\mathtt{M}]\!]} A \twoheadrightarrow B$. Then, A and B share at least one propositional variable at the same depth. [41, p.116]

So to show that a logic **L** has the DRP, one only needs to show that **L** is verified by a wr-model structure $[\![\mathtt{M}]\!]$.

3 Connexive wr-model structures

Though the origins of what we now call connexive logic go back to the work of ancient and Medieval philosophers, Storrs McCall brought this topic to the attention of modern logicians in his [28]. There, the idea of connexive implication is attributed to Chrysippus, and the following passage from Sextus Empiricus is provided as a characterization of the idea of connexive implication [24, p.129]:

> And those who introduce the notion of connection say that a conditional is sound when the contradictory of its consequent is incompatible with its antecedent.

So the notion of connection is cashed out in terms of (in)compatibility. Where $\circ$ is a primitive connective representing compatibility, we have that $A \to B$ if and only if $\sim(A \circ \sim B)$. Philosophical stances regarding this notion of compatibility lead to interesting logical theses. For instance, (1) if every proposition is compatible with itself —as Nelson argues [35, pp. 440, 447]—, i.e. if $A \circ A$ for every A, then $\sim(A \to \sim A)$ for every A; and (2) if a proposition implies another one, then these are compatible —as McCall suggests in [1, p.435]—, i.e. if $(A \to B) \to (A \circ B)$ for every A and every B, then $(A \to B) \to \sim(A \to \sim B)$ for every A and every B. Indeed, besides their philosophical appeal, these theses are invalid in classical logic; so any logical system including such principles would be a contra-classical logic [19].

A simple, working definition of connexive logics, strongly supported by Wansing and some of his collaborators [57], [59], [58], [37], [36] after some ideas of McCall [29], [31], is the following:

Definition 3.1. A logic **L** with a conditional $>$ and a negation $-$ is *connexive* iff the following hold [57]:

(AT) *Aristotle's Thesis*: $\vdash_{\mathbf{L}} -(A > -A)$
(AT') *Variant of Aristotle's Thesis*: $\vdash_{\mathbf{L}} -(-A > A)$
(BT) *Boethius' Thesis*: $\vdash_{\mathbf{L}} (A > B) > -(A > -B)$
(BT') *Variant of Boethius' Thesis*: $\vdash_{\mathbf{L}} (A > -B) > -(A > B)$
(NSC) *Non-symmetry of the conditional*: $\nvdash_{\mathbf{L}} (A > B) > (B > A)$

That definition will be used in this paper for the sake of simplicity. Keep in mind, however, that things are not as simple as that definition suggests, since there may be good reasons to think that a logic is still connexive if (i) it satisfies some but not all of (AT), (AT'), (BT), (BT') [20], [15], [38]; or (ii) if instead or besides of Aristotle's and Boethius' theses some other contra-classical theorems are demanded [51], [26], [23]; or (iii) if more than one kind of negation or more than one kind of conditional can be involved in expressing connexive theses [39], [40], [14]; or (iv) if Aristotle's and Boethius' theses are to be restricted to special kinds of propositions [22]; or (v) if some requirements should be met at the metatheory [21]; or (vi) if some of the connexive theses should be expressed in rule form instead of implicational form [60]; or (vii) if instead of (NSC) some other schema expressing the non-symmetry of a conditional is demanded; etc.[2]

The connexive theses have an intuitive appeal, specially for relevant logicians since it seems that this idea of connection between antecedents and consequents in a sound implication is closely related to the concept of logical relevance. Evidence of this interest can be found in [46], [1, §29.8], [33], [43], [45, §2.4], [34], [47], [8], [50], [51], [52, §9.7–8], [27], for instance, all authored by prominent relevant logicians.

However, connexivity and logical relevance are not easy to combine. Though both belong to the broader family of sociative logics [50], relevant logics and connexive logics can bring undesirable results when put together. As shown by Routley and Montgomery [46], Mortensen [34], and the authors of [45], there are three recurrent problems in combining them:

1. *Contradictoriness*; e.g. **B** plus Aristotle's Thesis is a contradictory logic.
2. *Trivialization*; e.g. **R** plus Aristotle's Thesis is a trivial logic.
3. *Validating the negation of every implication*; e.g. **E** plus Aristotle validates $\sim(A \rightarrow B)$.

Out of the three main problems, the most pressing one is that of triviality. This result works for very strong relevant systems like **R** that include, besides the extensional conjunction $\wedge$, an intensional conjunction $\circ$ that residuates the relevant implication. So non-trivial systems combining logical relevance and

[2]See [59] for the most recent attempt at delimiting what connexive logic is and what terminology shoud be adopted. But proceed with caution, for the classifications and decisions made there are far from being established opinions.

connexivity are generally expected to be weaker than **R**. On the other side of the spectrum, the basic relevant logic **B**, of which **R** and every other standard relevant system in between is an extension, will deliver contradictory theorems if connexive theses are introduced; so if one wants to combine logical relevance and connexivity, one should be prepared to embrace contradictory logics. This is not to be feared by logicians studying connexivity, like Wansing, whose well-known system **C** and its extensions are contradictory, yet not relevant logics —see [37], for instance.

To avoid validating all negated implications, a common result in logics combining relevance and connexivity, like **EA** and **M3V** [34], [13], some alternatives are available, as noted in [54]. Following Tedder's analysis of the proof of $\sim(A \to B)$ in [45, pp.245-6], one could abandon Rule Contraposition, $\vdash A \to B \;\therefore \vdash \sim B \to \sim A$, or Rule Prefixing, $\vdash A \to B \;\therefore \vdash (C \to A) \to (C \to B)$, or Rule Suffixing, $\vdash A \to B \;\therefore \vdash (B \to C) \to (A \to C)$, or Rule Transitivity, $\vdash A \to B, \vdash B \to C \;\therefore \vdash A \to C$. Each of these options could be seen to be a high price to pay for relevant logicians.[3] But on the interest of combining logical relevance and connexivity without validating all negated implications, in what follows I will be abandoning Rule Suffixing.

Observe that (BT) and (BT') are conditional formulas in which the antecedent shares a variable with the consequent at the same depth.[4] That the connexive theses satisfy the DRP is an intriguing feature that invites us to explore whether deep relevant logics admit the connexive theses without trivializing and without delivering unappealing results like the negation of every implication —and that is what will be explored here.

In order to show how depth relevance can mix with connexivity, we now define **cCL** —a connexive variant of **CL**. The logical matrix $\mathtt{M}^{\mathbf{cCL}}$ is defined exactly as $\mathtt{M}^{\mathbf{CL}}$, except that $f_{\to}$ has the following table:

$\to$	0	1	2	3	4	5
*0	0	5	5	5	5	5
*1	0	4	5	5	5	5
*2	0	2	2	5	5	5
*3	0	3	5	3	5	5
*4	0	1	2	3	4	5
5	3	3	3	3	3	3

[3] As argued in [54], any of these alternatives implies losing tonicity features of $\to$, as per the gaggle theoretic framework [12]. Note that other options are available to stop the proof of $\sim(A \to B)$, like dropping Uniform Substitution.

[4] In fact, they satisfy a stronger property: all variables in the antecedent are shared in the consequent at the same depth. We may generalize this as the following property: A logic **L** has the *strict depth relevance property* (SDRP) iff, if $A \to B$ is a theorem of **L**, then all variables in A occur in B at the same depth. Clearly the SDRP is very strong, for it would invalidate most implicational theses of standard relevant logics, including —crucially for negation-consistent connexive logics— Simplification, $(A \wedge B) \to A$ and $(A \wedge B) \to B$. Studying systems with the SDRP can be appealing to connexive logicians but that will not be pursued here. Regarding strong versions of the DRP, see [25] and [18]. Other variable-sharing properties more related to connexivity can be found in [17].

Remark 3.1. To obtain (AT), (AT'), (BT) and (BT'), one can adopt the well-known technique of modifying the rows for non-designated antecedents so that the value of the conditional is some designated value which is a fixed point of the negation. In our case, the $(5, x)$ row (for $0 \leq x \leq 5$) of the table for the **CL** conditional is modified so as to output 3 instead of 0.

Remark 3.2. To make sure that $\sim(A \to B)$ will not be valid using a connexive and relevant conditional, at least one of the values in the column for the highest of the designated values has to be a designated value which is not a fixed point of negation. So the $(x, 0)$ column (for $0 \leq x \leq 5$) has value 0 at least once. This is no modification to the conditional of **CL** though.

Remark 3.3. Having value 3 instead of 0 in column $(x, 0)$ for $2 \leq x \leq 4$ would validate Aristotle's Second Thesis, $\vdash\sim((A \to B) \wedge (\sim A \to B))$, but invalidate Conjunctive Syllogism, $\vdash ((A \to B) \wedge (B \to C)) \to (A \to C)$, and Rule Prefixing.

These remarks should make it evident that there are many ways to obtain a connexive version of **CL**; after all, the only modification we made to the conditional from **CL** is that mentioned in Remark 3.1. The most evident alternative would be that of using 2 instead of 3 in the places where we have made modifications.[5] But other combinations are possible and we leave this for future research. Based on the conditional of **cCL**, we have the following tables for the biconditional and the compatibility connective, defined as $A \circ B \overset{df}{=} \sim(A \to \sim B)$.

$\leftrightarrow$	0	1	2	3	4	5
*0	0	5	5	5	5	5
*1	5	4	5	5	5	5
*2	5	5	2	5	5	5
*3	5	5	5	3	5	5
*4	5	5	5	5	4	5
5	5	5	5	5	5	3

$\circ$	0	1	2	3	4	5
*0	0	0	0	0	0	5
*1	0	0	0	0	1	5
*2	0	0	2	0	2	5
*3	0	0	0	3	3	5
*4	0	1	2	3	4	5
5	3	3	3	3	3	3

The logic **cCL** is defined through the following axiomatization:

A1. $\vdash A \to A$
A2. $\vdash (A \wedge B) \to A$
A3. $\vdash (A \wedge B) \to B$
A4. $\vdash ((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
A5. $\vdash ((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
A6. $\vdash ((A \to B) \wedge (B \to C)) \to (A \to C)$
A7. $\vdash (A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
A8. $\vdash A \vee (A \to B)$
A9. $\vdash ((A \to A) \to B) \to B$
A10. $\vdash (A \to (A \to B)) \to (A \to B)$

[5] Using 2 instead of 3 produces the same results I report in what follows, so I suspect these are equivalent ways to define **cCL**.

A11. $\vdash (A \wedge (A \to B)) \to B$
A12. $\vdash (\sim B \wedge (A \to B)) \to\sim A$
A13. $\vdash\sim\sim A \to A$
A14. $\vdash A\vee \sim A$
A15. $\vdash (A \to\sim A) \to\sim A$
A16. $\vdash ((A \to B) \wedge (A \to\sim B)) \to\sim A$
A17. $\vdash (A \to\sim B) \to\sim (A \to B)$
R1. $\vdash A, \vdash A \to B \ \therefore\vdash B$
R2. $\vdash A, \vdash B \ \therefore\vdash A \wedge B$
R3. $\vdash A \to\sim B \ \therefore\vdash B \to\sim A$
R4. $\vdash A \to B \ \therefore \ (C \to A) \to (C \to B)$

Proposition 3.1. The following are valid in $\mathtt{M}^{\mathbf{cCL}}$:

1. *Identity*: $\vdash A \to A$
2. *Simplification*: $\vdash (A \wedge B) \to A$; and $\vdash (A \wedge B) \to B$
3. *Addition*: $\vdash A \to (A \vee B)$; and $\vdash B \to (A \vee B)$
4. $\wedge$-*Composition*: $\vdash ((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
5. $\vee$-*Composition*: $\vdash ((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
6. *Distribution*: $\vdash (A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
7. *Conjunctive Syllogism*: $\vdash ((A \to B) \wedge (B \to C)) \to (A \to C)$
8. *Rule Transitivity*: $\vdash A \to B, \vdash B \to C \ \therefore\vdash A \to C$
9. *Rule Prefixing*: $\vdash A \to B \ \therefore\vdash (C \to A) \to (C \to B)$
10. *Double Negation laws*: $\vdash\sim\sim A \to A$ and $\vdash A \to\sim\sim A$
11. *De Morgan laws*: $\vdash\sim (A\wedge B) \leftrightarrow (\sim A\vee \sim B)$ and $\vdash\sim (A\vee B) \leftrightarrow (\sim A\wedge \sim B)$
12. *Rule Contraposition*: $\vdash A \to\sim B \ \therefore\vdash B \to\sim A$ and $\vdash A \to B \ \therefore\vdash\sim B \to\sim A$
13. *Modus Ponens*: $\vdash A, \vdash A \to B \ \therefore\vdash B$
14. *Modus Tollens*: $\vdash\sim B, \vdash A \to B \ \therefore\vdash\sim A$
15. *Pseudo-Modus Ponens*: $\vdash (A \wedge (A \to B)) \to B$
16. *Pseudo-Modus Tollens*: $\vdash (\sim B \wedge (A \to B)) \to\sim A$
17. *Clavius*: $\vdash (A \to\sim A) \to\sim A$
18. *Reductio*: $\vdash ((A \to B) \wedge (A \to\sim B)) \to\sim A$
19. *Specialized Assertion*: $\vdash ((A \to A) \to B) \to B$
20. *Contraction*: $\vdash (A \to (A \to B)) \to (A \to B)$
21. *RM3-axiom*: $\vdash A \vee (A \to B)$
22. (AT), (AT'), (BT) and (BT')
23. *Abelard's First Principle*: $\vdash\sim ((A \to B) \wedge (A \to\sim B))$
24. *cRM3-axiom*: $\vdash A\vee \sim (A \to B)$
25. *Disjunctive rules*: $\vdash C \vee A, \vdash C \vee (A \to B) \ \therefore\vdash C \vee B$; and $\vdash C \vee A \ \therefore\vdash C\vee \sim (A \to\sim A)$

Proof. By inspection on $\mathtt{M}^{\mathbf{cCL}}$.[6] □

[6] I verified this by writing a simple Python program available at `https://github.com/Fernando-Cano-Jorge/CL-cCL/tree/main`

In Proposition 3.1, theses 1.–9. are core principles of the positive fragment of most relevant systems while 10.–12. are basic principles for negation. Note that theses 15.–19. are implicative formulas that violate depth relevance, and also that 20. goes against strong paraconsistency; we will return to these issues later on. Finally, 22.–24. are connexive theses and 25. are disjunctive rules considered by Brady in his deep relevant systems.

Proposition 3.2. The following are invalid in $\mathtt{M}^{\mathbf{cCL}}$:

1. *Prefixing*: $\vdash (A \to B) \to ((C \to A) \to (C \to B))$
2. *Suffixing*: $\vdash (A \to B) \to ((B \to C) \to (A \to C))$
3. *Rule Suffixing*: $\vdash (A \to B) \therefore \vdash (B \to C) \to (A \to C)$
4. *Rule Affixing*: $\vdash A \to B, \vdash C \to D \therefore \vdash (B \to C) \to (A \to D)$
5. *Disjunctive Rule Affixing*: $\vdash E \vee (A \to B), \vdash E \vee (C \to D) \therefore \vdash E \vee ((B \to C) \to (A \to D))$
6. *Self-Distribution*: $\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$
7. *Assertion*: $\vdash A \to ((A \to B) \to B)$
8. *Permutation*: $\vdash (A \to (B \to C)) \to (B \to (A \to C))$
9. *Mingle*: $\vdash A \to (A \to A)$
10. *Contraposition*: $\vdash (A \to\sim B) \to (B \to\sim A)$ and $\vdash (A \to B) \to (\sim B \to\sim A)$
11. *Peirce's Law*: $\vdash ((A \to B) \to A) \to A$
12. *Monotonicity*: $\vdash (A \to C) \to ((A \wedge B) \to C)$
13. *Residuation*: $\vdash (A \to (B \to C)) \to ((A \circ B) \to C)$ and its converse
14. *Rule Residuation*: $\vdash A \to (B \to C) \therefore \vdash (A \circ B) \to C$ and its converse
15. *Negated conditional*: $\vdash\sim (A \to B)$
16. *Symmetry of the conditional*: $\vdash (A \to B) \to (B \to A)$
17. *Aristotle's Second Thesis*: $\vdash\sim ((A \to B) \wedge (\sim A \to B))$
18. *Abelard's Second Principle*: $\vdash\sim (A \to\sim B)$
19. *Converse of (BT)*: $\vdash\sim (A \to\sim B) \to (A \to B)$
20. *Converse of (BT')*: $\vdash\sim (A \to B) \to (A \to\sim B)$

Proof. The following are counterexamples to the respective principles:

1. $I(A) = 0$, $I(B) = 0$, $I(C) = 5$.
2. $I(A) = 0$, $I(B) = 0$, $I(C) = 1$.
3. $I(A) = 5$, $I(B) = 0$, $I(C) = 0$.
4. $I(A) = 5$, $I(B) = 0$, $I(C) = 0$, $I(D) = 0$.
5. $I(A) = 5$, $I(B) = 0$, $I(C) = 0$, $I(D) = 0$, $I(E) = 5$.
6. $I(A) = 0$, $I(B) = 1$, $I(C) = 0$.
7. $I(A) = 0$, $I(B) = 1$.
8. $I(A) = 5$, $I(B) = 0$, $I(C) = 0$.
9. $I(A) = 1$.
10. $I(A) = 0$, $I(B) = 5$ and $I(A) = 0$, $I(B) = 0$, respectively.
11. $I(A) = 2$, $I(B) = 3$.
12. $I(A) = 0$, $I(B) = 5$, $I(C) = 0$.

13. $I(A) = 5$, $I(B) = 0$, $I(C) = 2$ and $I(A) = 0$, $I(B) = 5$, $I(C) = 0$, respectively.
14. $I(A) = 5$, $I(B) = 0$, $I(C) = 2$ and $I(A) = 0$, $I(B) = 5$, $I(C) = 0$, respectively.
15. $I(A) = 0$, $I(B) = 0$.
16. $I(A) = 1$, $I(B) = 0$.
17. $I(A) = 1$, $I(B) = 0$.
18. $I(A) = 0$, $I(B) = 5$
19. $I(A) = 0$, $I(B) = 1$.
20. $I(A) = 0$, $I(B) = 1$.

□

Proposition 3.3. $\mathtt{M}^{\mathbf{cCL}}$ is a wr-matrix.

Proof. Let $x_i = 2$, $x_r = 3$ and $x_F = 5$. Then conditions 1.–3. of Definition 2.11 are satisfied. □

Corollary 3.1. **cCL** has the VSP.

Proof. By Theorem 2.2 and Proposition 3.3. □

Corollary 3.2. $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$ is a wr-model structure.

Proof. By Proposition 3.3 and Definition 2.12. In all interpretations, we will fix $x_T = 3$. □

Corollary 3.3. If $\models_{[\![\mathtt{M}^{\mathbf{cCL}}]\!]} A \twoheadrightarrow B$, then A and B share at least one propositional variable at the same depth.

Proof. From Corollary 3.2 and Theorem 2.3. □

Thus, none of the well-known paradoxes of implication are valid in **cCL** and one can reasonably claim that it is a relevant logic, besides being connexive. However, it certainly is not a deep relevant logic nor a strong paraconsistent logic. And even though **cCL** is not exactly a connexive extension of **B**, because it lacks Rule Affixing, it happens to be a contradictory logic.

Definition 3.2. A logic **L** is *contradictory* iff $\vdash_{\mathbf{L}} A$ and $\vdash_{\mathbf{L}} \sim A$.

Proposition 3.4. **cCL** is a contradictory logic.

Proof. The following derivation, using **cCL**-valid principles, establishes the result:

1. $\vdash (p \wedge \sim p) \rightarrow p$ [instance of Simplification]
2. $\vdash \sim p \rightarrow \sim (p \wedge \sim p)$ [Rule Contraposition, 1]
3. $\vdash (p \wedge \sim p) \rightarrow \sim p$ [instance of Simplification]
4. $\vdash (p \wedge \sim p) \rightarrow \sim (p \wedge \sim p)$ [Rule Transitivity, 3 and 2]

So $\vdash_{\mathbf{cCL}} (p\wedge \sim p) \rightarrow\sim (p\wedge \sim p)$. On the other hand, $\vdash_{\mathbf{cCL}}\sim ((p\wedge \sim p) \rightarrow\sim (p\wedge \sim p))$ is an instance of (AT). □

I will not dwell further into the properties of **cCL**. This system is introduced only because its logical matrix is suitable to build a hierarchical model structure for a deep relevant connexive logic. Such logic, called **cDR**, is defined through the following axiomatization:

A1. $\vdash A \twoheadrightarrow A$
A2. $\vdash (A \wedge B) \twoheadrightarrow A$
A3. $\vdash (A \wedge B) \twoheadrightarrow B$
A4. $\vdash ((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow C)) \twoheadrightarrow (A \twoheadrightarrow (B \wedge C))$
A5. $\vdash ((A \twoheadrightarrow C) \wedge (B \twoheadrightarrow C)) \twoheadrightarrow ((A \vee B) \twoheadrightarrow C)$
A6. $\vdash ((A \twoheadrightarrow B) \wedge (B \twoheadrightarrow C)) \twoheadrightarrow (A \twoheadrightarrow C)$
A7. $\vdash (A \wedge (B \vee C)) \twoheadrightarrow ((A \wedge B) \vee (A \wedge C))$
A8. $\vdash\sim\sim A \twoheadrightarrow A$
A9. $\vdash A\vee \sim A$
A10. $\vdash (A \twoheadrightarrow\sim B) \twoheadrightarrow\sim (A \twoheadrightarrow B)$
R1. $\vdash A, \vdash A \twoheadrightarrow B \ \therefore \vdash B$
R2. $\vdash A, \vdash B \ \therefore \vdash A \wedge B$
R3. $\vdash A \twoheadrightarrow\sim B \ \therefore \vdash B \twoheadrightarrow\sim A$
R4. $\vdash A \twoheadrightarrow B \ \therefore \vdash (C \twoheadrightarrow A) \twoheadrightarrow (C \twoheadrightarrow B)$

Note the similarities between Brady's axiomatization of $\mathbf{DR^d}$ and that of **cDR**.[7] A nice feature of this similarity is that many useful theorems and derived rules of $\mathbf{DR^d}$ carry over to **cDR**, like $\vdash A \twoheadrightarrow\sim\sim A$, and $\vdash A \twoheadrightarrow B \ \therefore \vdash\sim B \twoheadrightarrow\sim A$, and $\vdash A \twoheadrightarrow B, \vdash B \twoheadrightarrow C \ \therefore \vdash A \twoheadrightarrow C$, as well as Addition, De Morgan Laws, the Principle of Non-Contradiction and Modus Tollens, to name a few that will be used below. To establish the connexive theses in **cDR**, first we see that (BT) follows from A10., which is just (BT'):

1. $\vdash (A \twoheadrightarrow\sim B) \twoheadrightarrow\sim (A \twoheadrightarrow B)$ [A10]
2. $\vdash (A \twoheadrightarrow B) \twoheadrightarrow\sim (A \twoheadrightarrow\sim B)$ [R3, 1]

Then (AT) follows from (BT):

1. $\vdash (A \twoheadrightarrow A) \twoheadrightarrow\sim (A \twoheadrightarrow\sim A)$ [(BT)]
2. $\vdash A \twoheadrightarrow A$ [A1]
3. $\vdash\sim (A \twoheadrightarrow\sim A)$ [R1 2,1]

and (AT') follows from (BT'):

1. $\vdash (\sim A \twoheadrightarrow\sim A) \twoheadrightarrow\sim (\sim A \twoheadrightarrow A)$ [A10]

[7] As noted by Ed Mares in conversation, **cDR** also resembles Routley's **DK**. The most important difference between them is that **DK** has Contraposition in arrow form and it also has Rule Affixing, $\vdash A \rightarrow B, \vdash C \rightarrow D \ \therefore \vdash (B \rightarrow C) \rightarrow (A \rightarrow D)$, while **cDR** does not. Indeed, all axioms but not all rules of **B** are included in **cDR**.

2. $\vdash\sim A \twoheadrightarrow\sim A$ [A1]
3. $\vdash\sim(\sim A \twoheadrightarrow A)$ [R1 2,1]

Moreover, following nomenclature in [15], **cDR** is Abelardian (but not ultra-Abelardian [13], for $\sim(A \twoheadrightarrow\sim B)$ does not hold), since Abelard's First Principle is derivable thus:

1. $\vdash ((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow (A \twoheadrightarrow B)$ [A2]
2. $\vdash (A \twoheadrightarrow B) \twoheadrightarrow\sim(A \twoheadrightarrow\sim B)$ [(BT)]
3. $\vdash ((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow\sim(A \twoheadrightarrow\sim B)$ [Rule Transitivity 1,2]
4. $\vdash ((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow (A \twoheadrightarrow\sim B)$ [A3]
5. $\vdash (A \twoheadrightarrow\sim B) \twoheadrightarrow\sim(A \twoheadrightarrow B)$ [(BT')]
6. $\vdash ((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow\sim(A \twoheadrightarrow B)$ [Rule Transitivity 4,5]
7. $\vdash (((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow\sim(A \twoheadrightarrow B)) \wedge (((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow\sim(A \twoheadrightarrow\sim B))$ [R2 6,3]
8. $((((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow\sim(A \twoheadrightarrow B)) \wedge (((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow\sim(A \twoheadrightarrow\sim B))) \twoheadrightarrow (((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow (\sim(A \twoheadrightarrow B) \wedge \sim(A \twoheadrightarrow\sim B)))$ [A4]
9. $\vdash ((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow (\sim(A \twoheadrightarrow B) \wedge \sim(A \twoheadrightarrow\sim B))$ [R1 7,8]
10. $\vdash (\sim(A \twoheadrightarrow B) \wedge \sim(A \twoheadrightarrow\sim B)) \twoheadrightarrow\sim((A \twoheadrightarrow B) \vee (A \twoheadrightarrow\sim B))$ [De Morgan Laws]
11. $\vdash ((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow\sim((A \twoheadrightarrow B) \vee (A \twoheadrightarrow\sim B))$ [Rule Transitivity 9,10]
12. $\vdash ((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow ((A \twoheadrightarrow B) \vee (A \twoheadrightarrow\sim B))$ [Theorem $\vdash (C \wedge D) \twoheadrightarrow (C \vee D)$]
13. $\vdash (((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow ((A \twoheadrightarrow B) \vee (A \twoheadrightarrow\sim B))) \wedge (((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow\sim((A \twoheadrightarrow B) \vee (A \twoheadrightarrow\sim B)))$ [R2 12,11]
14. $\vdash ((((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow ((A \twoheadrightarrow B) \vee (A \twoheadrightarrow\sim B))) \wedge (((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow\sim((A \twoheadrightarrow B) \vee (A \twoheadrightarrow\sim B)))) \twoheadrightarrow (((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow (((A \twoheadrightarrow B) \vee (A \twoheadrightarrow\sim B)) \wedge \sim((A \twoheadrightarrow B) \vee (A \twoheadrightarrow\sim B))))$ [A4]
15. $\vdash ((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B)) \twoheadrightarrow (((A \twoheadrightarrow B) \vee (A \twoheadrightarrow\sim B)) \wedge \sim((A \twoheadrightarrow B) \vee (A \twoheadrightarrow\sim B)))$ [R1 13,14]
16. $\vdash\sim(((A \twoheadrightarrow B) \vee (A \twoheadrightarrow\sim B)) \wedge \sim((A \twoheadrightarrow B) \vee (A \twoheadrightarrow\sim B)))$ [Theorem $\vdash\sim(C \wedge \sim C)$]
17. $\vdash\sim((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow\sim B))$ [Modus Tollens 11,12]

To get $\mathbf{cDR^d}$, add the following disjunctive rule to **cDR**:

R5. $\vdash C \vee A, \vdash C \vee (A \twoheadrightarrow B) \therefore \vdash C \vee B$

Note that the disjunctive rule $\vdash C \vee A \therefore \vdash C \vee \sim(A \twoheadrightarrow\sim A)$ is superfluous in $\mathbf{cDR^d}$ since $\vdash_{\mathbf{cDR}} C \vee \sim(A \twoheadrightarrow\sim A)$ is derivable from $\vdash_{\mathbf{cDR}}\sim(A \twoheadrightarrow\sim A)$ and $\vdash_{\mathbf{cDR}} B \twoheadrightarrow (A \vee B)$ using R1. On the other hand, since the disjunctive Rule Affixing is invalid in $\mathtt{M}^{\mathbf{cCL}}$, and therefore also invalid in $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$, it follows from the arguments we will give below that it is not a rule of $\mathbf{cDR^d}$.

Since we aim to define a family of deep relevant connexive logics using wr-model structures, we use Brady's technique and define, in addition to T, some subsets of K in $\mathtt{M}^{\mathbf{cCL}}$: $T^* = \{0\}$, $a = \{0,1,2\}$, and $a^* = \{0,1,3\}$. We also redefine clause 6.3 in Definition 2.13 thus:

6.3.' for $j = \omega$,

6.3a $I_\omega(A \twoheadrightarrow B) \in T$ iff $I_j(A \rightarrow B) \in T$ for all j $(0 \leq j \leq \omega)$
6.3b $I_\omega(A \twoheadrightarrow B) \in T^*$ iff $I_j(A \rightarrow B) \in T^*$ for all j $(0 \leq j \leq \omega)$
6.3c $I_\omega(A \twoheadrightarrow B) \in a$ iff $I_j(A \rightarrow B) \in a$ for all j $(0 \leq j \leq \omega)$
6.3d $I_\omega(A \twoheadrightarrow B) \in a^*$ iff $I_j(A \rightarrow B) \in a^*$ for all j $(0 \leq j \leq \omega)$

Lemma 3.1. For all i $(0 \leq i \leq \omega)$,

(i) (a) $I_i(\sim A) \in T \Leftrightarrow I_i(A) \notin T^*$
(b) $I_i(\sim A) \in T^* \Leftrightarrow I_i(A) \notin T$
(c) $I_i(\sim A) \in a \Leftrightarrow I_i(A) \notin a^*$
(d) $I_i(\sim A) \in a^* \Leftrightarrow I_i(A) \notin a$

(ii) (a) $I_i(A \wedge B) \in T \Leftrightarrow I_i(A) \in T$ and $I_i(B) \in T$
(b) $I_i(A \wedge B) \in T^* \Leftrightarrow I_i(A) \in T^*$ and $I_i(B) \in T^*$
(c) $I_i(A \wedge B) \in a \Leftrightarrow I_i(A) \in a$ and $I_i(B) \in a$
(d) $I_i(A \wedge B) \in a^* \Leftrightarrow I_i(A) \in a^*$ and $I_i(B) \in a^*$

(iii) (a) $I_i(A \vee B) \in T \Leftrightarrow I_i(A) \in T$ or $I_i(B) \in T$
(b) $I_i(A \vee B) \in T^* \Leftrightarrow I_i(A) \in T^*$ or $I_i(B) \in T^*$
(c) $I_i(A \vee B) \in a \Leftrightarrow I_i(A) \in a$ or $I_i(B) \in a$
(d) $I_i(A \vee B) \in a^* \Leftrightarrow I_i(A) \in a^*$ or $I_i(B) \in a^*$

(iv) (a)

$$\begin{aligned} I_i(A \rightarrow B) \in T \Leftrightarrow & I_i(A) \in T \Rightarrow I_i(B) \in T, \text{ and} \\ & I_i(A) \in T^* \Rightarrow I_i(B) \in T^*, \text{ and} \\ & I_i(A) \in a \Rightarrow I_i(B) \in a, \text{ and} \\ & I_i(A) \in a^* \Rightarrow I_i(B) \in a^* \end{aligned}$$

(b)

$$I_i(A \rightarrow B) \in T^* \Leftrightarrow I_i(A) \in T \text{ and } I_i(B) \in T^*$$

(c)

$$\begin{aligned} I_i(A \rightarrow B) \in a \Leftrightarrow & I_i(A) \in T \text{ and } I_i(B) \in T^*, \text{ or} \\ & I_i(A) \in T \backslash a^* \text{ and } I_i(B) \in a \end{aligned}$$

(d)

$$\begin{aligned} I_i(A \rightarrow B) \in a^* \Leftrightarrow & I_i(A) \notin T, \text{ or} \\ & I_i(B) \in T^*, \text{ or} \\ & I_i(A) \in T \backslash a \text{ and } I_i(B) \in a^* \end{aligned}$$

Proof. By inspection of $\mathtt{M}^{\mathbf{cCL}}$.[8] □

Now we use the previous lemma to prove the analogue of Brady's Theorem 2 in [4].

Theorem 3.1. $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$ verifies $\mathbf{cDR^d}$.

Proof. Using Lemma 3.1. For each $\mathbf{cDR^d}$ axiom we need to prove it is valid in $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$; in particular, for axioms of the form $A \twoheadrightarrow B$, we need to prove that, for all valuations v and all j $(0 \leq j \leq \omega)$,

(1) $I_j(A) \in T \Rightarrow I_j(B) \in T$
(2) $I_j(A) \in T^* \Rightarrow I_j(B) \in T^*$
(3) $I_j(A) \in a \Rightarrow I_j(B) \in a$
(4) $I_j(A) \in a^* \Rightarrow I_j(B) \in a^*$

and for each $\mathbf{cDR^d}$ rule we need to prove that it preserves $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$-validity.

[*Ad* A1.]

Trivial.

[*Ad* A2.]

(I) To prove $I_j(A \wedge B) \in T \Rightarrow I_j(A) \in T$.

Assume $I_j(A \wedge B) \in T$ for arbitrary j $(0 \leq j \leq \omega)$. Then $I_j(A) \in T$ and $I_j(B) \in T$, so in particular $I_j(A) \in T$.

(II) To prove $I_j(A \wedge B) \in T^* \Rightarrow I_j(A) \in T^*$.

Similar to (I).

(III) To prove $I_j(A \wedge B) \in a \Rightarrow I_j(A) \in a$.

Similar to (I).

(IV) To prove $I_j(A \wedge B) \in a^* \Rightarrow I_j(A) \in a^*$.

Similar to (I).

[*Ad* A3.]

As for A2.

[8]Note that $\Rightarrow$ and $\Leftrightarrow$ are, respectively, material implication and material equivalence. These are not connectives in the language of $\mathbf{cDR^d}$ but belong to the meta-theory in which the result is proved. This mirrors Brady's approach to proving properties of $\mathbf{DR^d}$. Indeed, using classical logic at the level of the meta-theory while, at the same time, championing some non-classical logic can be seen as inappropriate, philosophically speaking —a charge of hypocrisy; this is why Weber's project [63, Ch.3] aims to erase the meta-theory vs. object-theory distinction and try to have a logical system that can speak about itself instead of relying on classical logic to do this. Here I will carry on without considering this objection. On the matter of using classical meta-theory in paraconsistent logics see [53].

[*Ad* A4.]

(I) To prove $I_j((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow C)) \in T \Rightarrow I_j(A \twoheadrightarrow (B \wedge C)) \in T$.

- (a) For $j = 0$, as $I_0(A \twoheadrightarrow (B \wedge C)) = 3$, $I_0(A \twoheadrightarrow (B \wedge C)) \in T$.
- (b) For $0 < j < \omega$, assume $I_j((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow C)) \in T$. Then $I_j(A \twoheadrightarrow B) \in T$ and $I_j(A \twoheadrightarrow C) \in T$, so $I_{j-1}(A \rightarrow B) \in T$ and $I_{j-1}(A \rightarrow C) \in T$. We have then (i) $I_{j-1}(A) \in T \Rightarrow I_{j-1}(B) \in T$ and $I_{j-1}(A) \in T^* \Rightarrow I_{j-1}(B) \in T^*$ and $I_{j-1}(A) \in a \Rightarrow I_{j-1}(B) \in a$ and $I_{j-1}(A) \in a^* \Rightarrow I_{j-1}(B) \in a^*$; and (ii) $I_{j-1}(A) \in T \Rightarrow I_{j-1}(C) \in T$ and $I_{j-1}(A) \in T^* \Rightarrow I_{j-1}(C) \in T^*$ and $I_{j-1}(A) \in a \Rightarrow I_{j-1}(C) \in a$ and $I_{j-1}(A) \in a^* \Rightarrow I_{j-1}(C) \in a^*$. So it follows that $I_{j-1}(A) \in T \Rightarrow I_{j-1}(B \wedge C) \in T$ and $I_{j-1}(A) \in T^* \Rightarrow I_{j-1}(B \wedge C) \in T^*$ and $I_{j-1}(A) \in a \Rightarrow I_{j-1}(B \wedge C) \in a$ and $I_{j-1}(A) \in a^* \Rightarrow I_{j-1}(B \wedge C) \in a^*$. Therefore $I_{j-1}(A \rightarrow (B \wedge C)) \in T$ so $I_j(A \twoheadrightarrow (B \wedge C)) \in T$.
- (c) For $j = \omega$, assume $I_\omega((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow C)) \in T$. Then, for all j, $I_j(A \rightarrow B) \in T$ and $I_j(A \rightarrow C) \in T$, so, as in (I.b), $I_j(A \rightarrow (B \wedge C)) \in T$, whence $I_\omega(A \twoheadrightarrow (B \wedge C)) \in T$.

(II) To prove $I_j((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow C)) \in T^* \Rightarrow I_j(A \twoheadrightarrow (B \wedge C)) \in T^*$.

- (a) For $j = 0$, since $I_0(A \twoheadrightarrow B) = I_0(A \twoheadrightarrow C) = I_0((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow C)) = 3$, we have $I_0((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow C)) \notin T^*$.
- (b) For $0 < j < \omega$, assume $I_j((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow C)) \in T^*$. Then $I_j(A \twoheadrightarrow B) \in T^*$ and $I_j(A \twoheadrightarrow C) \in T^*$, so $I_{j-1}(A \rightarrow B) \in T^*$ and $I_{j-1}(A \rightarrow C) \in T^*$. We have then (i) $I_{j-1}(A) \in T$ and $I_{j-1}(B) \in T^*$; and (ii) $I_{j-1}(A) \in T$ and $I_{j-1}(C) \in T^*$. So it follows that $I_{j-1}(A) \in T$ and $I_{j-1}(B \wedge C) \in T^*$, whence $I_{j-1}(A \rightarrow (B \wedge C)) \in T^*$, so $I_j(A \twoheadrightarrow (B \wedge C)) \in T^*$.
- (c) For $j = \omega$, as in (II.b).

(III) To prove $I_j((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow C)) \in a \Rightarrow I_j(A \twoheadrightarrow (B \wedge C)) \in a$.

- (a) For $j = 0$, since $I_0(A \twoheadrightarrow B) = I_0(A \twoheadrightarrow C) = I_0((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow C)) = 3$, we have $I_0((A \twoheadrightarrow B) \wedge I(A \twoheadrightarrow C)) \notin a$.
- (b) For $0 < j < \omega$, assume $I_j((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow C)) \in a$. Then $I_j(A \twoheadrightarrow B) \in a$ and $I_j(A \twoheadrightarrow C) \in a$, so $I_{j-1}(A \rightarrow B) \in a$ and $I_{j-1}(A \rightarrow C) \in a$. Then we have, on the one hand, (i) $I_{j-1}(A) \in T$ and $I_{j-1}(B) \in T^*$, or (ii) $I_{j-1}(A) \in T \backslash a^*$ and $I_i(B) \in a$; and, on the other hand, (iii) $I_{j-1}(A) \in T$ and $I_{j-1}(C) \in T^*$, or (iv) $I_{j-1}(A) \in T \backslash a^*$ and $I_{j-1}(C) \in a$. If (i) and (iii), then $I_{j-1}(A) \in T$ and $I_{j-1}(B \wedge C) \in T^*$, so $I_{j-1}(A \rightarrow (B \wedge C)) \in a$, whence $I_j(A \twoheadrightarrow (B \wedge C)) \in a$. If (i) and (iv), then $I_{j-1}(A) \in T \backslash a^*$ and $I_{j-1}(B \wedge C) \in a$, so $I_{j-1}(A \rightarrow (B \wedge C)) \in a$, whence $I_j(A \twoheadrightarrow (B \wedge C)) \in a$ —and similarly, if (ii) and (iii), or if (ii) and (iv). So $I_j(A \twoheadrightarrow (B \wedge C)) \in a$ in all cases.
- (c) For $j = \omega$, as in (III.b).

(IV) To prove $I_j((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow C)) \in a^* \Rightarrow I_j(A \twoheadrightarrow (B \wedge C)) \in a^*$.

- (a) For $j = 0$, since $I_0(A \twoheadrightarrow (B \wedge C)) = 3$, we have $I_0(A \twoheadrightarrow (B \wedge C)) \in a^*$.

(b) For $0 < j < \omega$, assume $I_j((A \twoheadrightarrow B) \wedge (A \twoheadrightarrow C)) \in a^*$. Then $I_j(A \twoheadrightarrow B) \in a^*$ and $I_j(A \twoheadrightarrow C) \in a^*$, so $I_{j-1}(A \rightarrow B) \in a^*$ and $I_{j-1}(A \rightarrow C) \in a^*$. Then we have, on the one hand, (i) $I_{j-1}(A) \notin T$, or (ii) $I_{j-1}(B) \in T^*$, or (iii) $I_{j-1}(A) \in T \backslash a$ and $I_{j-1}(B) \in a^*$; and, on the other hand, (iv) $I_{j-1}(A) \notin T$, or (v) $I_{j-1}(C) \in T^*$, or (vi) $I_{j-1}(A) \in T \backslash a$ and $I_{j-1}(C) \in a^*$. If (i) and (iv), $I_{j-1}(A \rightarrow (B \wedge C)) \in a^*$ follows immediately, so $I_j(A \twoheadrightarrow (B \wedge C)) \in a^*$ —and similarly if (i) and (v), or if (i) and (vi), or if (ii) and (iv). If (ii) and (v), then $I_{j-1}(B \wedge C) \in T^*$, so $I_{j-1}(A \rightarrow (B \wedge C)) \in a^*$ and then $I_j(A \twoheadrightarrow (B \wedge C)) \in a^*$. If (ii) and (vi), then $I_{j-1}(A) \in T \backslash a$ and $I_{j-1}(B \wedge C) \in a^*$, so $I_{j-1}(A \rightarrow (B \wedge C)) \in a^*$ and then $I_j(A \twoheadrightarrow (B \wedge C)) \in a^*$ —and similarly if (iii) and (v), or if (iii) and (vi). Cases (iii) and (iv) cannot occur together. So $I_j(A \twoheadrightarrow (B \wedge C)) \in a^*$ in all cases.

(c) For $j = \omega$, as in (IV.b).

[*Ad* A5.]

(I) To prove $I_j((A \twoheadrightarrow C) \wedge (B \twoheadrightarrow C)) \in T \Rightarrow I_j((A \vee B) \twoheadrightarrow C) \in T$.

(a) For $j = 0$, since $I_0((A \vee B) \twoheadrightarrow C) = 3$, then $I_0((A \vee B) \twoheadrightarrow C) \in T$.

(b) For $0 < j < \omega$, assume $I_j((A \twoheadrightarrow C) \wedge (B \twoheadrightarrow C)) \in T$. Then $I_j(A \twoheadrightarrow C) \in T$ and $I_j(B \twoheadrightarrow C) \in T$, so $I_{j-1}(A \rightarrow C) \in T$ and $I_{j-1}(B \rightarrow C) \in T$. Then we have, on the one hand, $I_{j-1}(A) \in T \Rightarrow I_{j-1}(C) \in T$ and $I_{j-1}(A) \in T^* \Rightarrow I_{j-1}(C) \in T^*$ and $I_{j-1}(A) \in a \Rightarrow I_{j-1}(C) \in a$ and $I_{j-1}(A) \in a^* \Rightarrow I_{j-1}(C) \in a^*$; and, on the other hand, $I_{j-1}(B) \in T \Rightarrow I_{j-1}(C) \in T$ and $I_{j-1}(B) \in T^* \Rightarrow I_{j-1}(C) \in T^*$ and $I_{j-1}(B) \in a \Rightarrow I_{j-1}(C) \in a$ and $I_{j-1}(B) \in a^* \Rightarrow I_{j-1}(C) \in a^*$. If $I_{j-1}(A \vee B) \in T$, then $I_{j-1}(A) \in T$ or $I_{j-1}(B) \in T$, so $I_{j-1}(C) \in T$. If $I_{j-1}(A \vee B) \in T^*$, then $I_{j-1}(A) \in T^*$ or $I_{j-1}(B) \in T^*$, so $I_{j-1}(C) \in T^*$. If $I_{j-1}(A \vee B) \in a$, then $I_{j-1}(A) \in a$ or $I_{j-1}(B) \in a$, so $I_{j-1}(C) \in a$. If $I_{j-1}(A \vee B) \in a^*$, then $I_{j-1}(A) \in a^*$ or $I_{j-1}(B) \in a^*$, so $I_{j-1}(C) \in a^*$. Therefore, $I_{j-1}((A \vee B) \rightarrow C) \in T$, so $I_j((A \vee B) \twoheadrightarrow C) \in T$.

(c) For $j = \omega$, as in (I.b).

(II) To prove $I_j((A \twoheadrightarrow C) \wedge (B \twoheadrightarrow C)) \in T^* \Rightarrow I_j((A \vee B) \twoheadrightarrow C) \in T^*$.

(a) For $j = 0$, since $I_0(A \twoheadrightarrow C) = I_0(B \twoheadrightarrow C) = I_0((A \twoheadrightarrow C) \wedge (B \twoheadrightarrow C)) = 3$, $I_0((A \twoheadrightarrow C) \wedge (B \twoheadrightarrow C)) \notin T^*$.

(b) For $0 < j < \omega$, assume $I_j((A \twoheadrightarrow C) \wedge (B \twoheadrightarrow C)) \in T^*$. Then $I_j(A \twoheadrightarrow C) \in T^*$ and $I_j(B \twoheadrightarrow C) \in T^*$, so $I_{j-1}(A \rightarrow C) \in T^*$ and $I_{j-1}(B \rightarrow C) \in T^*$. Then $I_{j-1}(A) \in T$ and $I_{j-1}(B) \in T$ and $I_{j-1}(C) \in T^*$, so $I_{j-1}(A \vee B) \in T$ whence $I_{j-1}((A \vee B) \rightarrow C) \in T^*$ and so $I_j((A \vee B) \twoheadrightarrow C) \in T^*$.

(c) For $j = \omega$, as in (II.b).

(III) To prove $I_j((A \twoheadrightarrow C) \wedge (B \twoheadrightarrow C)) \in a \Rightarrow I_j((A \vee B) \twoheadrightarrow C) \in a$.

(a) For $j = 0$, since $I_0(A \twoheadrightarrow C) = I_0(B \twoheadrightarrow C) = I_0((A \twoheadrightarrow C) \wedge (B \twoheadrightarrow C)) = 3$, $I_0((A \twoheadrightarrow C) \wedge (B \twoheadrightarrow C)) \notin a$.

(b) For $0 < j < \omega$, assume $I_j((A \twoheadrightarrow C) \wedge (B \twoheadrightarrow C)) \in a$. Then $I_j(A \twoheadrightarrow C) \in a$ and $I_j(B \twoheadrightarrow C) \in a$, so $I_{j-1}(A \rightarrow C) \in a$ and $I_{j-1}(B \rightarrow C) \in a$. We then have, on the one hand, (i) $I_{j-1}(A) \in T$ and $I_{j-1}(C) \in T^*$, or (ii) $I_{j-1}(A) \in T \backslash a^*$ and $I_{j-1}(C) \in a$; and, on the other hand, (iii) $I_{j-1}(B) \in T$ and $I_{j-1}(C) \in T^*$, or (iv) $I_{j-1}(B) \in T \backslash a^*$ and $I_{j-1}(C) \in a$. If (i) and (iii), $I_{j-1}(A \vee B) \in T$ and $I_{j-1}(C) \in T^*$, so $I_{j-1}((A \vee B) \rightarrow C) \in a$, whence $I_j((A \vee B) \twoheadrightarrow C) \in a$ —and similarly if (i) and (iv), or if (ii) and (iii). If (ii) and (iv), $I_{j-1}(A \vee B) \in T \backslash a^*$ and $I_{j-1}(C) \in a$, so $I_{j-1}((A \vee B) \rightarrow C) \in a$ whence $I_j((A \vee B) \twoheadrightarrow C) \in a$. So in all cases $I_j((A \vee B) \twoheadrightarrow C) \in a$.

(c) For $j = \omega$, as in (III.b).

(IV) To prove $I_j((A \twoheadrightarrow C) \wedge (B \twoheadrightarrow C)) \in a^* \Rightarrow I_j((A \vee B) \twoheadrightarrow C) \in a^*$.

(a) For $j = 0$, since $I_0((A \vee B) \twoheadrightarrow C) = 3$, $I_j((A \vee B) \twoheadrightarrow C) \in a^*$.

(b) For $0 < j < \omega$, assume $I_j((A \twoheadrightarrow C) \wedge (B \twoheadrightarrow C)) \in a^*$. Then $I_j(A \twoheadrightarrow C) \in a^*$ and $I_j(B \twoheadrightarrow C) \in a^*$, so $I_{j-1}(A \rightarrow C) \in a^*$ and $I_{j-1}(B \rightarrow C) \in a^*$. We then have, on the one hand, (i) $I_{j-1}(A) \notin T$, or (ii) $I_{j-1}(C) \in T^*$, or (iii) $I_{j-1}(A) \in T \backslash a$ and $I_{j-1}(C) \in a^*$; and, on the other hand, (iv) $I_{j-1}(B) \notin T$, or (v) $I_{j-1}(C) \in T^*$, or (vi) $I_{j-1}(B) \in T \backslash a$ and $I_{j-1}(C) \in a^*$. If (i) and (iv), $I_{j-1}(A \vee B) \notin T$, so $I_{j-1}((A \vee B) \rightarrow C) \in a^*$, whence $I_j((A \vee B) \twoheadrightarrow C) \in a^*$. If (i) and (v), since $I_{j-1}(C) \in T^*$, $I_{j-1}((A \vee B) \rightarrow C) \in a^*$, whence $I_j((A \vee B) \twoheadrightarrow C) \in a^*$ —and similarly if (ii) and (iv), or if (ii) and (v), or if (ii) and (vi), or if (iii) and (v). If (i) and (vi), $I_{j-1}(A \vee B) \in T$ since $I_{j-1}(B) \in T$, and $I_{j-1}(A \vee B) \notin a$ since $I_{j-1}(A) \notin a$ and $I_{j-1}(B) \notin a$, so $I_{j-1}(A \vee B) \in T \backslash a$ and then $I_{j-1}((A \vee B) \rightarrow C) \in a^*$ whence $I_j((A \vee B) \twoheadrightarrow C) \in a^*$ —and similarly if (iii) and (iv), or if (iii) and (vi). So in all cases $I_j((A \vee B) \twoheadrightarrow C) \in a^*$.

(c) For $j = \omega$, as in (IV.b).

[*Ad* A6.]

(I) To prove $I_j((A \twoheadrightarrow B) \wedge (B \twoheadrightarrow C)) \in T \Rightarrow I_j(A \twoheadrightarrow C) \in T$.

(a) For $j = 0$, since $I_0(A \twoheadrightarrow C) = 3$, $I_0(A \twoheadrightarrow C) \in T$.

(b) For $0 < j < \omega$, assume $I_j((A \twoheadrightarrow B) \wedge (B \twoheadrightarrow C)) \in T$. Then $I_j(A \twoheadrightarrow B) \in T$ and $I_j(A \twoheadrightarrow C) \in T$, so $I_{j-1}(A \rightarrow B) \in T$ and $I_{j-1}(B \rightarrow C) \in T$. We have then (i) $I_{j-1}(A) \in T \Rightarrow I_{j-1}(B) \in T$ and $I_{j-1}(A) \in T^* \Rightarrow I_{j-1}(B) \in T^*$ and $I_{j-1}(A) \in a \Rightarrow I_{j-1}(B) \in a$ and $I_{j-1}(A) \in a^* \Rightarrow I_{j-1}(B) \in a^*$; and (ii) $I_{j-1}(B) \in T \Rightarrow I_{j-1}(C) \in T$ and $I_{j-1}(B) \in T^* \Rightarrow I_{j-1}(C) \in T^*$ and $I_{j-1}(B) \in a \Rightarrow I_{j-1}(C) \in a$ and $I_{j-1}(B) \in a^* \Rightarrow I_{j-1}(C) \in a^*$. So $I_{j-1}(A) \in T \Rightarrow I_{j-1}(C) \in T$ and $I_{j-1}(A) \in T^* \Rightarrow I_{j-1}(C) \in T^*$ and $I_{j-1}(A) \in a \Rightarrow I_{j-1}(C) \in a$ and $I_{j-1}(A) \in a^* \Rightarrow I_{j-1}(C) \in a^*$, whence $I_{j-1}(A \rightarrow C) \in T$ so $I_j(A \twoheadrightarrow C) \in T$.

(c) For $j = \omega$, as in (I.b).

(II) To prove $I_j((A \twoheadrightarrow B) \wedge (B \twoheadrightarrow C)) \in T^* \Rightarrow I_j(A \twoheadrightarrow C) \in T^*$.

(a) For $j = 0$, since $I_0(A \twoheadrightarrow B) = I_0(B \twoheadrightarrow C) = I_0((A \twoheadrightarrow B) \wedge (B \twoheadrightarrow C)) = 3$, we have $I_0((A \twoheadrightarrow B) \wedge (B \twoheadrightarrow C)) \notin T^*$.

(b) For $0 < j < \omega$, assume $I_j((A \twoheadrightarrow B) \wedge (B \twoheadrightarrow C)) \in T^*$. Then $I_j(A \twoheadrightarrow B) \in T^*$ and $I_j(B \twoheadrightarrow C) \in T^*$, so $I_{j-1}(A \to B) \in T^*$ and $I_{j-1}(B \to C) \in T^*$. We have then (i) $I_{j-1}(A) \in T$ and $I_{j-1}(B) \in T^*$; and (ii) $I_{j-1}(B) \in T$ and $I_{j-1}(C) \in T^*$. So it follows that $I_{j-1}(A \to C) \in T^*$, whence $I_j(A \twoheadrightarrow C) \in T^*$.

(c) For $j = \omega$, as in (II.b).

(III) To prove $I_j((A \twoheadrightarrow B) \wedge (B \twoheadrightarrow C)) \in a \Rightarrow I_j(A \twoheadrightarrow C)) \in a$.

(a) For $j = 0$, since $I_0(A \twoheadrightarrow B) = I_0(B \twoheadrightarrow C) = I_0((A \twoheadrightarrow B) \wedge (B \twoheadrightarrow C)) = 3$, we have $I_0((A \twoheadrightarrow B) \wedge (B \twoheadrightarrow C)) \notin a$.

(b) For $0 < j < \omega$, assume $I_j((A \twoheadrightarrow B) \wedge (B \twoheadrightarrow C)) \in a$. Then $I_j(A \twoheadrightarrow B) \in a$ and $I_j(B \twoheadrightarrow C) \in a$, so $I_{j-1}(A \to B) \in a$ and $I_{j-1}(B \to C) \in a$. Then we have, on the one hand, (i) $I_{j-1}(A) \in T$ and $I_{j-1}(B) \in T^*$, or (ii) $I_{j-1}(A) \in T \backslash a^*$ and $I_{j-1}(B) \in a$; and, on the other hand, (iii) $I_{j-1}(B) \in T$ and $I_{j-1}(C) \in T^*$, or (iv) $I_{j-1}(B) \in T \backslash a^*$ and $I_{j-1}(C) \in a$. If (i) and (iii), then $I_{j-1}(A) \in T$ and $I_{j-1}(C) \in T^*$, so $I_{j-1}(A \to C) \in a$, whence $I_j(A \twoheadrightarrow C) \in a$ —and similarly if (ii) and (iii). Cases (i) and (iv) cannot occur together. If (ii) and (iv), then $I_{j-1}(A) \in T \backslash a^*$ and $I_{j-1}(C) \in a$, so $I_{j-1}(A \to C) \in a$, whence $I_j(A \twoheadrightarrow C) \in a$. So $I_j(A \twoheadrightarrow C) \in a$ in all cases.

(c) For $j = \omega$, as in (III.b).

(IV) To prove $I_j((A \twoheadrightarrow B) \wedge (B \twoheadrightarrow C)) \in a^* \Rightarrow I_j(A \twoheadrightarrow C) \in a^*$.

(a) For $j = 0$, since $I_0(A \twoheadrightarrow C) = 3$, we have $I_0(A \twoheadrightarrow C) \in a^*$.

(b) For $0 < j < \omega$, assume $I_j((A \twoheadrightarrow B) \wedge (B \twoheadrightarrow C)) \in a^*$. Then $I_j(A \twoheadrightarrow B) \in a^*$ and $I_j(B \twoheadrightarrow C) \in a^*$, so $I_{j-1}(A \to B) \in a^*$ and $I_{j-1}(B \to C) \in a^*$. We then have, on the one hand, (i) $I_{j-1}(A) \notin T$, or (ii) $I_{j-1}(B) \in T^*$, or (iii) $I_{j-1}(A) \in T \backslash a$ and $I_{j-1}(B) \in a^*$; and, on the other hand, (iv) $I_{j-1}(B) \notin T$, or (v) $I_{j-1}(C) \in T^*$, or (vi) $I_{j-1}(B) \in T \backslash a$ and $I_{j-1}(C) \in a^*$. If (i) and (iv), $I_{j-1}(A \to C) \in a^*$ follows immediately, so $I_j(A \twoheadrightarrow C) \in a^*$ —and similarly if (i) and (v), or if (i) and (vi). Cases (ii) and (iv) cannot occur together —and similarly if (ii) and (vi), or if (iii) and (iv). If (ii) and (v), it immediately follows that $I_{j-1}(A \to C) \in a^*$, so $I_j(A \twoheadrightarrow C) \in a^*$ —and similarly if (iii) and (v). If (iii) and (vi), then $I_{j-1}(A) \in T \backslash a$ and $I_{j-1}(C) \in a^*$, so $I_{j-1}(A \to C) \in a^*$, whence $I_j(A \twoheadrightarrow C) \in a^*$. So $I_j(A \twoheadrightarrow C) \in a^*$ in all cases.

(c) For $j = \omega$, as in (IV.b).

[*Ad* A7.]

(I) To prove $I_j(A \wedge (B \vee C)) \in T \Rightarrow I_j((A \wedge B) \vee (A \wedge C)) \in T$.

Assume $I_j(A \wedge (B \vee C)) \in T$ for arbitrary j $(0 \leq j \leq \omega)$. Then $I_j(A) \in T$ and $I_j(B \vee C) \in T$, i.e. $I_j(B) \in T$ or $I_j(C) \in T$, so

$[I_j(A) \in T$ and $I_j(B) \in T]$ or $[I_j(A) \in T$ and $I_j(C) \in T]$ and therefore $I_j((A \wedge B) \vee (A \wedge C)) \in T$.

(II) To prove $I_j(A \wedge (B \vee C)) \in T^* \Rightarrow I_j((A \wedge B) \vee (A \wedge C)) \in T^*$.

Similar to (I).

(III) To prove $I_j(A \wedge (B \vee C)) \in a \Rightarrow I_j((A \wedge B) \vee (A \wedge C)) \in a$.

Similar to (I).

(IV) To prove $I_j(A \wedge (B \vee C)) \in a^* \Rightarrow I_j((A \wedge B) \vee (A \wedge C)) \in a^*$.

Similar to (I).

[*Ad* A8.]

(I) To prove $I_j({\sim\sim} A) \in T \Rightarrow I_j(A) \in T$.

Assume $I_j({\sim\sim} A) \in T$ for arbitrary j $(0 \leq j \leq \omega)$. Then $I_j({\sim} A) \notin T^*$ so $I_j(A) \in T$.

(II) To prove $I_j({\sim\sim} A) \in T^* \Rightarrow I_j(A) \in T^*$.

Assume $I_j({\sim\sim} A) \in T^*$ for arbitrary j $(0 \leq j \leq \omega)$. Then $I_j({\sim} A) \notin T$ so $I_j(A) \in T^*$.

(III) To prove $I_j({\sim\sim} A) \in a \Rightarrow I_j(A) \in a$.

Assume $I_j({\sim\sim} A) \in a$ for arbitrary j $(0 \leq j \leq \omega)$. Then $I_j({\sim} A) \notin a^*$ so $I_j(A) \in a$.

(IV) To prove $I_j({\sim\sim} A) \in a^* \Rightarrow I_j(A) \in a^*$.

Assume $I_j({\sim\sim} A) \in a^*$ for arbitrary j $(0 \leq j \leq \omega)$. Then $I_j({\sim} A) \notin a$ so $I_j(A) \in a^*$.

[*Ad* A9.]

$I_\omega(A \vee \sim A) \in T \Leftrightarrow [I_\omega(A) \in T$ or $I_\omega(\sim A) \in T] \Leftrightarrow [I_\omega(A) \in T$ or $I_\omega(A) \notin T^*]$. So A9. is valid in $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$.

[*Ad* A10.]

(I) To prove $I_j(A \twoheadrightarrow {\sim} B) \in T \Rightarrow I_j({\sim}(A \twoheadrightarrow B)) \in T$.

(a) For $j = 0$, as $I_0(A \twoheadrightarrow B) = 3 = I_0({\sim}(A \twoheadrightarrow B))$, then $I_0({\sim}(A \twoheadrightarrow B)) \in T$.

(b) For $0 < j < \omega$, assume $I_j(A \twoheadrightarrow {\sim} B) \in T$. Then $I_{j-1}(A \rightarrow {\sim} B) \in T$, so, in particular, $I_{j-1}(A) \in T \Rightarrow I_{j-1}({\sim} B) \in T$. Assume $I_{j-1}(A \rightarrow B) \in T^*$. Then $I_{j-1}(A) \in T$ and $I_{j-1}(B) \in T^*$; it follows that $I_{j-1}({\sim} B) \in T$, but then $I_{j-1}(B) \notin T^*$, a contradiction. Therefore $I_{j-1}(A \rightarrow B) \notin T^*$ so $I_{j-1}({\sim}(A \rightarrow B)) \in T$, whence $I_j({\sim}(A \twoheadrightarrow B)) \in T$.

(c) For $j = \omega$, as in (I.b).

(II) To prove $I_j(A \twoheadrightarrow {\sim} B) \in T^* \Rightarrow I_j({\sim}(A \twoheadrightarrow B)) \in T^*$.

(a) For $j = 0$, since $I_0(A \twoheadrightarrow\sim B) = 3$, $I_0(A \twoheadrightarrow\sim B) \notin T^*$.

(b) For $0 < j < \omega$, assume $I_j(A \twoheadrightarrow\sim B) \in T^*$. So $I_{j-1}(A \rightarrow\sim B) \in T^*$, whence $I_{j-1}(A) \in T$ and $I_{j-1}(\sim B) \in T^*$, i.e. $I_{j-1}(B) \notin T$. Assume $I_{j-1}(A \rightarrow B) \in T$; then, in particular, $I_{j-1}(A) \in T \Rightarrow I_{j-1}(B) \in T$, so $I_{j-1}(B) \in T$ follows, a contradiction. Therefore $I_{j-1}(A \rightarrow B) \notin T$, so $I_{j-1}(\sim(A \rightarrow B)) \in T^*$, whence $I_j(\sim(A \twoheadrightarrow B)) \in T^*$.

(c) For $j = \omega$, as in (II.b).

(III) To prove $I_j(A \twoheadrightarrow\sim B) \in a \Rightarrow I_j(\sim(A \twoheadrightarrow B)) \in a$.

(a) For $j = 0$, since $I_0(A \twoheadrightarrow\sim B) = 3$, $I_0(A \twoheadrightarrow\sim B) \notin a$.

(b) For $0 < j < \omega$, assume $I_j(A \twoheadrightarrow\sim B) \in a$. So $I_{j-1}(A \rightarrow\sim B) \in a$, whence (i) $I_{j-1}(A) \in T$ and $I_{j-1}(\sim B) \in T^*$, i.e. $I_{j-1}(B) \notin T$, or (ii) $I_{j-1}(A) \in T\backslash a^*$ and $I_{j-1}(\sim B) \in a$, i.e. $I_{j-1}(B) \notin a^*$. Assume $I_{j-1}(A \rightarrow B) \in a^*$. Then (iii) $I_{j-1}(A) \notin T$, or (iv) $I_{j-1}(B) \in T^*$, or (v) $I_{j-1}(A) \in T\backslash a$ and $I_{j-1}(B) \in a^*$. If (i) and (iii), contradiction —and similarly if (i) and (iv), or if (i) and (v), or if (ii) and (iii), or if (ii) and (iv), or if (ii) and (v). Therefore $I_{j-1}(A \rightarrow B) \notin a^*$, so $I_{j-1}(\sim(A \rightarrow B)) \in a$, whence $I_j(\sim(A \twoheadrightarrow B)) \in a$.

(c) For $j = \omega$, as in (III.b).

(IV) To prove $I_j(A \twoheadrightarrow\sim B) \in a^* \Rightarrow I_j(\sim(A \twoheadrightarrow B)) \in a^*$.

(a) For $j = 0$, since $I_0(A \twoheadrightarrow B) = 3 = I_0(\sim(A \twoheadrightarrow B))$, $I_0(\sim(A \twoheadrightarrow B)) \in a^*$.

(b) For $0 < j < \omega$, assume $I_j(A \twoheadrightarrow\sim B) \in a^*$. So $I_{j-1}(A \rightarrow\sim B) \in a^*$, whence (i) $I_{j-1}(A) \notin T$, or (ii) $I_{j-1}(\sim B) \in T^*$, i.e. $I_{j-1}(B) \notin T$, or (iii) $I_{j-1}(A) \in T\backslash a$ and $I_{j-1}(\sim B) \in a^*$, i.e. $I_{j-1}(B) \notin a$. Assume $I_{j-1}(A \rightarrow B) \in a$. Then (iv) $I_{j-1}(A) \in T$ and $I_{j-1}(B) \in T^*$, or (v) $I_{j-1}(A) \in T\backslash a^*$ and $I_{j-1}(B) \in a$. If (i) and (iv), contradiction —and similarly if (i) and (v), or if (ii) and (iv), or if (ii) and (v), or if (iii) and (iv), or if (iii) and (v). Therefore, $I_{j-1}(A \rightarrow B) \notin a$, so $I_{j-1}(\sim(A \rightarrow B)) \in a^*$, whence $I_j(\sim(A \rightarrow B)) \in a^*$.

(c) For $j = \omega$, as in (IV.b).

[*Ad* R1.]

Let $\models_{[\![\mathtt{M}^{\mathbf{cCL}}]\!]} A$ and $\models_{[\![\mathtt{M}^{\mathbf{cCL}}]\!]} A \twoheadrightarrow B$. Then $I_\omega(A) \in T$ and $I_\omega(A \twoheadrightarrow B) \in T$, for all valuations v. Then, for arbitrary v, $I_\omega(A \rightarrow B) \in T$ so by Lemma 3.1 $I_\omega(A) \in T \Rightarrow I_\omega(B) \in T$. Therefore $I_\omega(B) \in T$ so $\models_{[\![\mathtt{M}^{\mathbf{cCL}}]\!]} B$. Hence R1. preserves $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$-validity.

[*Ad* R2.]

Let $\models_{[\![\mathtt{M}^{\mathbf{cCL}}]\!]} A$ and $\models_{[\![\mathtt{M}^{\mathbf{cCL}}]\!]} B$. Then $I_\omega(A) \in T$ and $I_\omega(B) \in T$ for all valuations v. Then, for arbitrary v, by Lemma 3.1, $I_\omega(A \wedge B) \in T$ so $\models_{[\![\mathtt{M}^{\mathbf{cCL}}]\!]} A \wedge B$. Hence R2. preserves $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$-validity.

[*Ad* R3.]

Let $\models_{[\![\mathtt{M}^{\mathbf{cCL}}]\!]} A \twoheadrightarrow\sim B$. Then $I_\omega(A \twoheadrightarrow\sim B) \in T$ for all valuations v. Then, for arbitrary v, $I_\omega(A \to\sim B) \in T$ so by Lemma 3.1 (i) $I_\omega(A) \in T \Rightarrow I_\omega(\sim B) \in T$ and therefore $I_\omega(\sim B) \notin T \Rightarrow I_\omega(A) \notin T$ so $I_\omega(B) \in T^* \Rightarrow I_\omega(\sim A) \in T^*$; (ii) $I_\omega(A) \in T^* \Rightarrow I_\omega(\sim B) \in T^*$ and therefore $I_\omega(\sim B) \notin T^* \Rightarrow I_\omega(A) \notin T^*$ so $I_\omega(B) \in T \Rightarrow I_\omega(\sim A) \in T$; (iii) $I_\omega(A) \in a \Rightarrow I_\omega(\sim B) \in a$ and therefore $I_\omega(\sim B) \notin a \Rightarrow I_\omega(A) \notin a$ so $I_\omega(B) \in a^* \Rightarrow I_\omega(\sim A) \in a^*$; and (iv) $I_\omega(A) \in a^* \Rightarrow I_\omega(\sim B) \in a^*$ and therefore $I_\omega(\sim B) \notin a^* \Rightarrow I_\omega(A) \notin a^*$ so $I_\omega(B) \in a \Rightarrow I_\omega(\sim A) \in a$. Hence $I_\omega(B \to\sim A) \in T$ for all v, so $I_\omega(B \twoheadrightarrow\sim A) \in T$ and then $\models_{[\![\mathtt{M}^{\mathbf{cCL}}]\!]} B \twoheadrightarrow\sim A$. So R3. preserves $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$-validity.

[*Ad* R4.]

Let $\models_{[\![\mathtt{M}^{\mathbf{cCL}}]\!]} A \twoheadrightarrow B$. Then $I_\omega(A \twoheadrightarrow B) \in T$ for all valuations v. Then, for arbitrary v, $I_\omega(A \to B) \in T$, so by Lemma 3.1 $I_\omega(A) \in T \Rightarrow I_\omega(B) \in T$, and $I_\omega(A) \in T^* \Rightarrow I_\omega(B) \in T^*$, and $I_\omega(A) \in a \Rightarrow I_\omega(B) \in a$, and $I_\omega(A) \in a^* \Rightarrow I_\omega(B) \in a^*$. We reason thus:

(a) If $I_\omega(C \twoheadrightarrow A) \in T$, then $I_\omega(C) \in T \Rightarrow I_\omega(A) \in T$, and $I_\omega(C) \in T^* \Rightarrow I_\omega(A) \in T^*$, and $I_\omega(C) \in a \Rightarrow I_\omega(A) \in a$, and $I_\omega(C) \in a^* \Rightarrow I_\omega(A) \in a^*$. Then $I_\omega(C) \in T \Rightarrow I_\omega(B) \in T$, and $I_\omega(C) \in T^* \Rightarrow I_\omega(B) \in T^*$, and $I_\omega(C) \in a \Rightarrow I_\omega(B) \in a$, and $I_\omega(C) \in a^* \Rightarrow I_\omega(B) \in a^*$. So $I_\omega(C \twoheadrightarrow B) \in T$.

(b) If $I_\omega(C \twoheadrightarrow A) \in T^*$, then $I_\omega(C) \in T$ and $I_\omega(A) \in T^*$, whence $I_\omega(B) \in T^*$, so $I_\omega(C \twoheadrightarrow B) \in T^*$.

(c) If $I_\omega(C \twoheadrightarrow A) \in a$, then (i) $I_\omega(C) \in T$ and $I_\omega(A) \in T^*$, or (ii) $I_\omega(C) \in T\backslash a^*$ and $I_\omega(A) \in a$. If (i), then $I_\omega(B) \in T^*$, so $I_\omega(C \twoheadrightarrow B) \in a$. If (ii), then $I_\omega(B) \in a$, so $I_\omega(C \twoheadrightarrow B) \in a$. Hence, $I_\omega(C \twoheadrightarrow B) \in a$.

(d) If $I_\omega(C \twoheadrightarrow A) \in a^*$, then (i) $I_\omega(C) \notin T$, or (ii) $I_\omega(A) \in T^*$, or (iii) $I_\omega(C) \in T\backslash a$ and $I_\omega(A) \in a^*$. If (i), then $I_\omega(C \twoheadrightarrow B) \in a^*$. If (ii), then $I_\omega(B) \in T^*$, so $I_\omega(C \twoheadrightarrow B) \in a^*$. If (iii), then $I_\omega(B) \in a^*$, so $I_\omega(C \twoheadrightarrow B) \in a^*$. Hence $I_\omega(C \twoheadrightarrow B) \in a^*$.

Therefore $I_\omega((C \twoheadrightarrow A) \to (C \twoheadrightarrow B)) \in T$, so $I_\omega((C \twoheadrightarrow A) \twoheadrightarrow (C \twoheadrightarrow B)) \in T$. Hence $\models_{[\![\mathtt{M}^{\mathbf{cCL}}]\!]} (C \twoheadrightarrow A) \twoheadrightarrow (C \twoheadrightarrow B)$. So R4. preserves $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$-validity.

[*Ad* R5.]

Let $\models_{[\![\mathtt{M}^{\mathbf{cCL}}]\!]} C \vee A$ and $\models_{[\![\mathtt{M}^{\mathbf{cCL}}]\!]} C \vee (A \twoheadrightarrow B)$. Then $I_\omega(C \vee A) \in T$ and $I_\omega(C \vee (A \twoheadrightarrow B))$ for all valuations v. So for arbitrary v, let $I_\omega(C) \notin T$. Then $I_\omega(A) \in T$ and $I_\omega(A \twoheadrightarrow B) \in T$. Then by the argument for R1., $I_\omega(B) \in T$, so $I_\omega(C \vee B) \in T$ and then $\models_{[\![\mathtt{M}^{\mathbf{cCL}}]\!]} C \vee B$. So R5. preserves $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$-validity.

So all axioms of $\mathbf{cDR^d}$ are valid in $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$ and all rules of $\mathbf{cDR^d}$ preserve $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$-validity. Therefore, $\mathbf{cDR^d}$ is verified by $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$. □

Theorem 3.2. $\mathbf{cDR^d}$ has the DRP.

Proof. By Corollary 3.3 and Theorem 3.1. □

Corollary 3.4. $\mathbf{cDR^d}$ has the VSP, the AP and the CAP.

Proof. By Theorem 3.2 and Theorem 2.1. □

Theorem 3.3. Let **X** and **Y** be systems containing "$\twoheadrightarrow$" such that all theorems of **X** are theorems of **Y**. Then, if **Y** satisfies the DRP, so does **X**.

Proof. Let $\vdash_{\mathbf{X}} A \twoheadrightarrow B$. Then $\vdash_{\mathbf{Y}} A \twoheadrightarrow B$ and hence A and B share a variable at the same depth. □

Corollary 3.5. **cDR** has the DRP.

Proof. Clearly **cDR** is theoremwise contained in $\mathbf{cDR^d}$ so by Theorem 3.3 the result follows. □

Corollary 3.6. **cDR** has the VSP, the AP and the CAP.

Proof. By Corollary 3.5 and Theorem 2.1. □

Note that having the DRP is sufficient to know that **cDR** and $\mathbf{cDR^d}$ are non-trivial logics, since no implicative formula violating the DRP will be provable. Thus depth relevance is a means through which one can avoid the most pressing problem of combining relevance and connexivity. Moreover, **cDR** and $\mathbf{cDR^d}$ do not validate the negation of every conditional, since their underlying matrix $\mathtt{M}^{\mathbf{cCL}}$ rules out that result. Still, as expected, we have that:

Proposition 3.5. **cDR** and $\mathbf{cDR^d}$ are contradictory logics.

Proof. As in the proof of Proposition 3.4. □

Finally, we provide some examples related to the strong paraconsistency of **cDR** and $\mathbf{cDR^d}$.

Example 3.1. The assignment given in Example 2.2 also works for $\mathbf{cDR^d}$ using $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$, so Pseudo-Modus Ponens is invalid in $\mathbf{cDR^d}$. Now let X be the formula $(p \twoheadrightarrow (p \twoheadrightarrow q)) \twoheadrightarrow (p \twoheadrightarrow q)$, an instance of Contraction. Let A be the antecedent of X; and let B be the consequent of X. We have $\deg(X) = 3$, $\deg(A) = 2$, $\deg(B) = 1$, $d[p_1, A] = 1$, $d[p_2, A] = 2 = d[q, A]$ and $d[p, B] = 1 = d[q, B]$. We use $[\![\mathtt{M}^{\mathbf{cCL}}]\!]$ to provide a valuation v_{m-d-1}, where $m = \deg(X)$ and d is the depth of each propositional variable in A or in B. By Lemma 2.1, this valuation can be extended to an interpretation I_{m-1} that assigns 2 to A and 3 to B, so that the value of X is 5, thus:

1. $v_{3-2-1}(p) = 2 = v_{3-2-1}(q)$
2. $v_{3-1-1}(p) = 4$ and $v_{3-1-1}(q) = 3$

Then $I_0(p \to q) = 2$ so $I_1(p \twoheadrightarrow q) = 2$ and then $I_1(p \to (p \twoheadrightarrow q)) = 2$ whence $I_2(p \twoheadrightarrow (p \twoheadrightarrow q)) = 2$; on the other hand, $I_1(p \to q) = 3$ so $I_2(p \twoheadrightarrow q) = 3$; then $I_2((p \twoheadrightarrow (p \twoheadrightarrow q)) \to (p \twoheadrightarrow q)) = 5$ so $I_3((p \twoheadrightarrow (p \twoheadrightarrow q)) \twoheadrightarrow (p \twoheadrightarrow q)) = 5$. So Contraction is invalid in $\mathbf{cDR^d}$.

4 Conclusions

By building on Brady's and Robles and Méndez's results, and by modifying Meyer's crystal lattice so as to get a connexive variant of it, a deep relevant connexive logic $\mathbf{cDR^d}$ was obtained. This system and those contained in it witness non-trivial, yet contradictory systems combining connexivity and logical relevance without validating undesirable theorems like $\sim(A \to B)$.

Many interesting questions arise from the work presented here. Regarding a semantics for $\mathbf{cDR^d}$, a natural question is whether a content containment semantics like that of $\mathbf{DJ^d}$ can be provided [6], [9]. While Brady gave a Routley-Meyer-style semantics for relevant logics containing Aristotle's and Boethius' theses in [8], such proposal will not work for $\mathbf{cDR^d}$ and its extensions, since the semantics given there applies to affixing systems and $\mathbf{cDR^d}$ is crucially non-affixing.

As for proof theory, it remains to be seen if $\mathbf{cDR^d}$, like $\mathbf{DJ^d}$, is gentzenizable or if it has a natural deduction system [5], [9]. Another open problem is if $\mathbf{cDR^d}$ is decidable, just as $\mathbf{DJ^d}$ is [9]. Indeed, such nice properties of $\mathbf{DJ^d}$ may not transfer to $\mathbf{cDR^d}$ as depth relevance did.

On the other hand, **cCL**, just as its non-connexive counterpart, can be very useful to logicians investigating the intersection of connexive logics and relevant logics. Relevant systems lacking the DRP but including the connexive theses can be shown to have the VSP through $\mathtt{M}^{\mathbf{cCL}}$, just as subsystems of **R** can be shown to have the VSP through $\mathtt{M}^{\mathbf{CL}}$. Related to such investigations, it can be worth exploring whether a negation-consistent relevant connexive logic can be designed by adopting the SDRP mentioned in footnote 4. Indeed, such a system would be quite weak, for it would invalidate most implicational theses, including Simplification. However, it may be more in line with the investigations of McCall [29], Angell [2] and Nelson [35].

But what about connexive naïve set theory? Following [64], one can show that all set theories based on connexive logics with Simplification are inconsistent. More specifically, Wiredu's argument shows that the typical Zermelo way out of Russell's paradox is unavailable for set theories based on connexive logics with Simplification; i.e. using the Separation axiom instead of the Unrestricted Comprehension axiom leads to contradictory theorems if one's set theory is based on a connexive logic with Simplification. The proof goes as follows:

1. $\vdash \forall z \exists y \forall x(x \in y \leftrightarrow (x \in z \wedge \varphi x))$ [Separation axiom]
2. $\vdash y \in y \leftrightarrow (y \in z \wedge \sim(y \in y))$ [From 1., x/y, y/y, z/z, $\varphi :=\sim (x \in x)$]
3. $\vdash (y \in y \to y \in z) \wedge (y \in y \to\sim (y \in y))$ [From 2., by definition of $\leftrightarrow$, Simplification and $\wedge$-Composition]
4. $\vdash y \in y \to\sim(y \in y)$ [From 3., Simplification]
5. $\vdash\sim(y \in y \to\sim(y \in y))$ [Aristotle's Thesis]
6. $\vdash (y \in y \to\sim(y \in y))\wedge \sim(y \in y \to\sim(y \in y))$ [From 4. and 5., Adjunction]

Simplification seems to be key in Wiredu's proof —just as in the proof of the inconsistency of **cDR** and $\mathbf{cDR^d}$. Oddly, he considers McCall's connexive logic

CC1 [29], where Simplification is invalid; he claims that the reason why Simplification is invalid is that $(A \wedge B) \rightarrow A$ cannot have contradictory substitutions for B as in $(p \wedge \sim p) \rightarrow p$ because, as per his interpretation of McCall's work, contradictions are inferentially innert in connexive settings.[9] But even with that restriction in place, since in the proof above no contradictory substitutions in Simplification are used, Wiredu's argument apparently shows that connexive set theories with Simplification are contradictory.[10] This result holds both for ZF set theories, i.e. those using Separation instead of Unrestricted Comprehension, and for naïve set theory:

1. $\vdash \exists y \forall x (x \in y \leftrightarrow \varphi)$ [Unrestricted Comprehension axiom]
2. $\vdash y \in y \leftrightarrow \sim (y \in y)$ [From 1., x/y, y/y, $\varphi := \sim (x \in x)$]
3. $\vdash y \in y \rightarrow \sim (y \in y)$ [From 2., Simplification]
4. $\vdash \sim (y \in y \rightarrow \sim (y \in y))$ [Aristotle's Thesis]
5. $\vdash (y \in y \rightarrow \sim (y \in y)) \wedge \sim (y \in y \rightarrow \sim (y \in y))$ [From 3. and 4., Adjunction]

Wiredu's diagnosis is that *it is not the "separation" requirement in Zermelo's axiom that is decisive in avoiding Russell's paradox but its truth functionality* [64, p.130]. And that may or not be so. While Wiredu's argument could discourage some connexive logicians like McCall, pursuing negation-consistency, it does not discourage those, like myself, that are comfortable working in inconsistent connexive logics. A contradictory, strongly paraconsistent logic like **cDR**, or rather $\mathbf{cDR^dQ}$, seems well suited for naïve set theory. Indeed, if naïve set theory is characterized by the Unrestricted Comprehension axiom, which in turn allows for the existence of inconsistent sets like R —or worse, like Routley's $\mathcal{Z} = \{x : x \notin \mathcal{Z}\}$—, why not have a contradictory logic as its foundation? It was once dreamed that the contradictory logic **DL** [44], named "dialectical logic", would be such a foundation.

Consider, then, $\mathbf{cDR^dQ}$, i.e. the first-order logic obtainded by adding the following axioms to $\mathbf{cDR^d}$ and by adjusting the language appropriately:

QA1. $\vdash (\forall x A) \twoheadrightarrow A[x/y]$, where y is free for x in A.
QA2. $\vdash \forall x (A \twoheadrightarrow B) \twoheadrightarrow (A \twoheadrightarrow \forall x B)$, where x is not free in A.

[9] In fact, for **CC1**, Wiredu's argument is intended to be a triviality proof, since **CC1** is explosive. At least that is so if logical consequence is the standard truth-preserving relation. Luis Estrada-González and Christian A. Romero-Rodríguez argue in [16] that one can avoid triviality as a consequence of Wiredu's argument by using Malinowski's q-consequence over **CC1**. They also note that Wiredu's account of the invalidity of Simplification in **CC1** is incorrect and that Simplification is not needed for the triviality proof, but Conjunction Elimination is, i.e. $\vdash A \wedge B \therefore \vdash A$ and $\vdash A \wedge B \therefore \vdash B$, which is shown to be invalid in **CC1** using q-consequence.

[10] Taking [16] into consideration, the correct conclusion should be that connexive set theories with Conjunction Elimination are contradictory, which is definitely a stronger conclusion. Contrast this situation with what happens in relevant naïve set theory, where there is a choice between having an inconsistent theory, like Routley's **DST** [42, pp.919-27], or a consistent one, like Brady's **CT** [9, §6], depending on whether the logic has (unrestrictedly) the Excluded Middle axiom. See [9, §5.4] and [11].

QA3. $\vdash \forall x(A \vee B) \twoheadrightarrow (A \vee \forall xB)$, where x is not free in A.
QA4. $\vdash A[x/y] \twoheadrightarrow \exists xA$, where y is free for x in A.
QA5. $\vdash \forall x(A \twoheadrightarrow B) \twoheadrightarrow (\exists xA \twoheadrightarrow B)$, where x is not free in B.
QA6. $\vdash (A \wedge \exists xB) \twoheadrightarrow \exists x(A \wedge B)$, where x is not free in A.
QR1. $\vdash A \mathrel{.\,.\!\!\vdash} \forall xA$

The addition of quantificational axioms does not affect depth relevance; axioms QA1.–QA6. and rule QR1. are standard across all relevant logics [9, p.38]. So let us see how $\mathbf{cDR^dQ}$ deals with Curry's paradox and its variants.

Clearly, the standard version of Curry's paradox, the one using Contraction, will not go through:

1. $\vdash \exists y \forall x(x \in y \leftrightarrow \varphi)$ [Unrestricted Comprehension axiom]
2. $\vdash y \in y \leftrightarrow (y \in y \rightarrow A)$ [From 1., x/y, y/y, $\varphi := x \in x \rightarrow A$]
3. $\vdash y \in y \rightarrow (y \in y \rightarrow A)$ [From 2., Simplification]
4. $\vdash (y \in y \rightarrow (y \in y \rightarrow A)) \rightarrow (y \in y \rightarrow A)$ [Instance of Contraction]
5. $\vdash y \in y \rightarrow A$ [From 3. and 4., Modus Ponens]
6. $\vdash (y \in y \rightarrow A) \rightarrow y \in y$ [From 2., Simplification]
7. $\vdash y \in y$ [From 5. and 6., Modus Ponens]
8. $\vdash A$ [From 5. and 7., Modus Ponens]

Neither will the following, depending on Pseudo-Modus Ponens:

1. $\vdash \exists y \forall x(x \in y \leftrightarrow \varphi)$ [Unrestricted Comprehension axiom]
2. $\vdash y \in y \leftrightarrow (y \in y \rightarrow A)$ [From 1., x/y, y/y, $\varphi := x \in x \rightarrow A$]
3. $\vdash (y \in y \wedge (y \in y \rightarrow A)) \rightarrow A$ [Instance of Pseudo-Modus Ponens]
4. $\vdash (y \in y \wedge y \in y) \rightarrow A$ [From 2. and 3., Substitution]
5. $\vdash y \in y \rightarrow (y \in y \wedge y \in y)$ [Instance of Idempotence of $\wedge$]
6. $\vdash y \in y \rightarrow A$ [From 5. and 4., Rule Transitivity]
7. $\vdash (y \in y \rightarrow A) \rightarrow y \in y$ [From 2., Simplification]
8. $\vdash y \in y$ [From 6. and 7., Modus Ponens]
9. $\vdash A$ [From 7. and 8., Modus Ponens]

Similar arguments like Slaney's [49] break apart as well, since those heavily depend on the stronger principle of Permutation, which is not even valid in **cCL**. Finally, there is another argument, by Routley et al [45, p.367], involving $\circ$ and both directions of Rule Residuation:

1. $\vdash \exists y \forall x(x \in y \leftrightarrow \varphi)$ [Unrestricted Comprehension axiom]
2. $\vdash y \in y \leftrightarrow ((y \in y \circ y \in y) \rightarrow A)$ [From 1., x/y, y/y, $\varphi := (x \in x \circ x \in x) \rightarrow A$]
3. $\vdash y \in y \rightarrow ((y \in y \circ y \in y) \rightarrow A)$ [From 2., Simplification]
4. $\vdash y \in y \rightarrow (y \in y \rightarrow (y \in y \circ y \in y))$ [From Rule Residuation and Identity[11]]

[11] Consider the instance of Identity $\vdash (y \in y \circ y \in y) \rightarrow (y \in y \circ y \in y)$; using Rule Residuation, i.e. $\vdash (A \circ B) \rightarrow C \mathrel{.\,.\!\!\vdash} A \rightarrow (B \rightarrow C)$, where $A/y \in y$, $B/y \in y$ and $C/y \in y \circ y \in y$, we have $\vdash y \in y \rightarrow (y \in y \rightarrow (y \in y \circ y \in y))$.

5. $\vdash y \in y \rightarrow ((y \in y \rightarrow (y \in y \circ y \in y)) \wedge ((y \in y \circ y \in y) \rightarrow A))$ [From 4. and 3., $\wedge$-Composition]
6. $\vdash y \in y \rightarrow (y \in y \rightarrow A)$ [From 5, Conjunctive Syllogism and Rule Transitivity]
7. $\vdash (y \in y \circ y \in y) \rightarrow A$ [From 6., Rule Residuation]
8. $\vdash y \in y$ [From 2. and 7., Simplification and Modus Ponens]
9. $\vdash y \in y \rightarrow A$ [From 8. and 6., Modus Ponens]
10. $\vdash A$ [From 8. and 9., Modus Ponens]

Due to the properties of $\circ$, this argument does not go through using $\mathbf{cDR^dQ}$. On the one hand, for any A, $\vdash A \circ A$, since by definition this is equivalent to Aristotle's thesis, $\vdash\sim (A \twoheadrightarrow\sim A)$; and, for any A and any B, $\vdash (A \twoheadrightarrow B) \twoheadrightarrow (A \circ B)$, which by definition is equivalent to Boethius' Thesis, $\vdash (A \twoheadrightarrow B) \twoheadrightarrow\sim (A \twoheadrightarrow\sim B)$. However, $\circ$ is not the residual of $\twoheadrightarrow$, as we saw in **cCL**, so the argument is blocked by rejecting steps 4 and 7. The fact that both directions of Residuation and Rule Residuation fail in **cCL** and $\mathbf{cDR^d}$ means that in these logics $\circ$ is *not a fusion connective*, as it is in **R**. The failure of Residuation principles, also known as Exportation and Importation (or, jointly, as Portation), suggests that $\circ$ is better interpreted as a *compatibility connective* in **cCL** and $\mathbf{cDR^d}$, like that addressed in Chrysippus' account of (connexive) implication.

In Weber's naïve set theory [63, Ch. 5], these versions of Curry's paradox are avoided by using a substructural logic, called `sub`**DLQ**, which (1) instead of an extensional conjunction $\wedge$ uses a non-idempotent intensional conjunction &, and (2) uses a relevant conditional $\rightarrow$ that does not validate Contraction to formalize set-theoretic axioms, and an irrelevant conditional $\Rightarrow$ that obeys the Deduction Theorem. Thus, he avoids the Contraction-Curry thanks to his relevant conditional; he avoids the Pseudo-Modus-Ponens-Curry by rejecting $(A \wedge (A \rightarrow B)) \rightarrow B$ while accepting $(A \& (A \Rightarrow B)) \Rightarrow B$; and he avoids the Residuation-Curry by dropping the idempotent conjunction $\wedge$ in favor of the non-idempotent intensional connective &.

In the case of $\mathbf{cDR^dQ}$, less machinery is needed to avoid the Curry paradoxes. Indeed, in $\mathbf{cDR^dQ}$ no Deduction Theorem will hold, since the rule Modus Ponens holds while Pseudo-Modus Ponens does not, even when phrased as $(A \circ (A \twoheadrightarrow B)) \twoheadrightarrow B$, which can clearly be seen to go against depth relevance. Weber wants some version of Pseudo-Modus Ponens so that a Deduction theorem is available and hence an argument can be made against the object-theory vs. meta-theory distinction. Clearly, $\mathbf{cDR^dQ}$ still has that distinction in place, for Modus Ponens is not valid in the language but over the language of $\mathbf{cDR^dQ}$; moreover, $\mathbf{cDR^dQ}$ has a classical meta-theory rather than a relevant and connexive one matching the object-theory.

In terms of mathematical strenght, a naïve set theory based on $\mathbf{cDR^dQ}$ would be weaker than Weber's, since the latter can resort to the irrelevant conditional where the relevant conditional falls short of demonstrative power, but would be similar to Routley's **DST** [42, §6], based on **DKQ**, or to Brady's based on $\mathbf{DJ^dQ}$. However, the connexive features of the logic can yield interesting differences, especially regarding subsets, complements and cardinality of powersets. In [30], a connexive algebra of classes is proposed where no class is contained in its complement —an idea that emulates Aristotle's thesis[12]—, and where the empty class is contained in itself but in no other set —and similarly for the universal class—, meaning that the number of subsets of any given set is $2^n - 1$ instead of the usual 2^n. While similar results may be expected using $\mathbf{cDR^dQ}$, the resulting theory would differ from McCall's since his is a consistent approach to connexive set theory.

Other topics addressed by Weber's work, like a suitable theory of identity that does not clash with the axiom of Extensionality, are needed to assess the suitability of a logic for naïve set theory. So this should also be taken into account regarding $\mathbf{cDR^dQ}$, for if no such theory of identity is available then the Hinnion-Libert paradox will trivialize naïve set theory [63, pp.122–3]. And if $\mathbf{cDR^dQ}$ cannot handle the Extensionality axiom (and substitution principles) without trivializing the theory, then one may consider dropping it in favour of some form of Leibniz equality law as in [55, §2.3].

Alas, connexive set theory, naïve or not, has seldom been investigated. So far, due to its ability to avoid Curry paradoxes, it seems that $\mathbf{cDR^dQ}$ may be a suitable foundation for connexive naïve set theory or a non-extensional version of it. But this ought to be properly explored in future work.

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