NON-CANONICAL MODELS OF RELEVANT LOGICS

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Abstract. We construct, in several ways, Fine-style frame models of the relevant logic **B** which verify exactly the theorems of **B** but differ from the canonical model in some of their properties: they contain multiple points representing the same (formal) **B** theory, or their points fail to represent some formal **B** theories (in fact, fail to represent almost all of them). We briefly discuss the implications of this for the adequacy of frame semantics for relevant logics.

1. Introduction

Relevant logics are logics that model a meaning for the implication connective $\rightarrow$ richer than the material semantics, rejecting some principles of classical and intuitionistic logic, especially weakening $((A \rightarrow (B \rightarrow A)))$. Almost all relevant logics studied so far can be seen as extensions of a simple logic called **B**, and semantics for them have been constructed by elaboration of semantic systems for **B**. Routley-Meyer models [1] are one such semantic system, in which models are sets of points, often called "worlds," and the semantics of $\rightarrow$ involves a ternary relation on worlds. This is seen as a generalization of the binary accessibility relation in Kripke models of modal logics (see [2]). Many philosophical interpretations of the ternary relation have been proposed [3][4][5]. Kit Fine demonstrated [6] a different semantics for **B**, also based on frames of points, but equipped with a binary operation $\circ$ instead of a ternary relation, which can be conceived of as "applying the rules of a theory (a point) to another theory (point)." This semantics has a very intuitive completeness proof: there is a canonical model whose points are all the *formal theories* with respect to **B**, which satisfies exactly the theorems of **B**. However, the definition of a Fine model involves complicated conditions, which seem designed to be satisfied just by this model. So far it has been plausible that the only Fine models satisfying exactly the theorems of **B** might be those in which each point represents (in a sense to be made precise) a unique formal theory, and every formal theory is represented by a point. Such models are so similar to the canonical model that they are almost a canonical model in disguise. (For a similar worry, see the conclusion of [7].) Thus the semantics would beg the question: we are using it to argue that **B** is interesting, but it would covertly presuppose there is something interesting about **B** formal theories.

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Nor can this problem be easily avoided by switching semantics: as shown in, e.g., [8], sec. 4, Fine and Routley-Meyer models can be constructed out of each other in a very natural way[1]. Thus if there is some artificiality about the points in all Fine models, there is also some artificiality about the filters of points in all Routley-Meyer models. The concern is merely easier to see from the Fine perspective, where the correspondence between theories and model components is more primary.

Recently, Restall and Standefer [9] introduced a new semantics for propositional relevant logics, in the same spirit as Routley-Meyer and Fine, using a multiary version of the Routley-Meyer relation. This relation is considerably more elegant than the ternary one, but does not claim to be any more than a generalization of the same idea, and these authors also prove a form of equivalence (see [9], Lemma 16). Indeed the completeness proof for $\mathbf{RW^+}$ in [9] uses a model derived from a canonical Routley-Meyer model. If the avoidance of our problem required a change of semantics, it may require a more severe change, perhaps content semantics as in [10] - but as it turns out it does not. We show that not all Fine models of exactly **B** do have the aforementioned 'artificial' properties, and we construct several counterexamples. We hope this makes the semantic argument for use of **B** more robust. At the same time, it raises a new concern about the semantics.

In the next section we give reasons to be interested in logic as theory-building system, which can motivate Fine's frames, but we hope that, because of formal similarities between semantic systems, our results will also be of interest to those who work with ternary (or multiary) accessibility relations, from other philosophical motivations.

After introducing the logic **B** in section 2, in section 3 we construct models with distinct "twin points" representing a single formal theory. First we do this with a construction that glues models together, then we give an alternate proof by taking a reduct from a canonical model. In section 4, we nonconstructively prove the existence of models such that there are "missing theories" not represented by any point in the model. In sections 5 and 6, we obtain models with missing theories in other ways, first by a pseudo-recursion-theoretic argument, and then by a simpler set-theoretic argument, which relies on set-theoretic assumptions.

2. Preliminaries

We assume knowledge of propositional logics, Hilbert proof systems and induction on formula complexity, length of proofs, etc. In later sections we will assume basic knowledge of first-order model theory, recursive functions, and set theory. Assume we have a fixed set of atomic propositional symbols At, and the connectives $\wedge, \vee, \neg, \rightarrow$.

2.1. B.

The logic **B** is given by the following Hilbert system:

Axioms:

(1) $A \rightarrow A$
(2) $(A \wedge B) \rightarrow A$
(3) $(A \wedge B) \rightarrow B$
(4) $A \rightarrow (A \vee B)$
(5) $B \rightarrow (A \vee B)$
(6) $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$
(7) $((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$
(8) $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$
(9) $\neg\neg A \rightarrow A$

Rules:

[1] For propositional logics. 'Fine's semantics' also refers to a way of modifying frame semantics for relevant logics to accommodate *quantifiers*, but that is not how we use the term here.

(1) $\dfrac{A \quad B}{A \wedge B}$

(2) $\dfrac{A \quad A \to B}{B}$

(3) $\dfrac{A \to B \quad C \to D}{(B \to C) \to (A \to D)}$

(4) $\dfrac{A \to \neg B}{B \to \neg A}$

2.2. **Semantics.** A *pre-model* is a 7-tuple $\langle T, P, \ell, \circ, \sqsubseteq, *, v\rangle$, such that:

- T is a set, $P \subseteq T$
- $\ell \in T$
- $\circ : T \times T \to T$
- $\sqsubseteq$ is a partial order on elements of T
- $* : P \to P$
- $v : T \to 2^{\mathsf{At}}$

For a pre-model $M = \langle T, P, \ell, \circ, \sqsubseteq, *, v\rangle$, we will refer to its components as T_M, P_M, ℓ_M, etc. as needed.

A pre-model is a *model* if it satisfies the following:

(1) If $s \sqsubseteq t$, then
  (a) $v(s) \subseteq v(t)$
  (b) $u \circ s \sqsubseteq u \circ t$ for all $u \in T$
  (c) $s \circ u \sqsubseteq t \circ u$ for all $u \in T$
(2) $\ell \circ t = t$ for all $t \in T$
(3) $p^{**} = p$
(4) If $p \sqsubseteq q$ for $p \in P$ and $q \in P$, then $q^* \sqsubseteq p^*$
(5) $v(t) = \bigcap_{t \subseteq p \in P} v(p)$
(6) If $t \circ u \subseteq p$, where $p \in P$, then there are $q, r \in P$ with $t \subseteq q$ and $u \subseteq r$ such that $t \circ r \subseteq p$ and $q \circ u \subseteq p$

We may refer to these points (T) as *theories*, P as *prime theories* or prime points, and ℓ as *the logic*. Define the verification relation $\models \subseteq (T \times \Phi)$ (which we may write $\models_M$) recursively on formulas (let Φ (or Φ_{At}) be the set of all formulas formed from At.)

(1) If $A \in \mathsf{At}$, $t \models A$ iff $A \in v(t)$
(2) $t \models A \wedge B$ iff $t \models A$ and $t \models B$
(3) $t \models A \vee B$ iff for every $p \in P$ where $t \sqsubseteq p$, $p \models A$ or $p \models B$
(4) $t \models A \to B$ iff for every $u \in T$ where $u \models A$, $t \circ u \models B$
(5) $t \models \neg A$ iff for every $p \in P$ where $t \sqsubseteq p$, $p^* \not\models A$

For formulas A, we say $M \models A$ (M is a model of A, M verifies/satisfies A) if $\ell_M \models_M A$, and A is *valid* if $M \models A$ for every model M. It is well-known that exactly the theorems of **B** are valid, but we will re-prove this in greater generality later in this paper.

Define the *derivation relation* $\vdash \subseteq (\mathcal{P}(\Phi) \times \Phi)$ for **B**, inductively, as follows. ($\mathcal{P}(\Phi)$ is the powerset of Φ.) It relates a set of formulas to a formula.

(1) If $A \in \Gamma$, $\Gamma \vdash A$
(2) If $\Gamma \vdash A$, $\Gamma \vdash B$, then $\Gamma \vdash A \wedge B$
(3) If $\Gamma \vdash B$ and $(B \to A) \in \mathbf{B}$, then $\Gamma \vdash A$

Let $\Gamma \subseteq \Phi$ be called a *formal theory* if, for all A such that $\Gamma \vdash A$, $A \in \Gamma$.

Call a set Γ *prime* if, whenever $A \vee B \in \Gamma$, either $A \in \Gamma$ or $B \in \Gamma$.

Define the *application* operation $\circ$: $\Gamma \circ \Delta = \{B \mid \exists A[A \in \Delta, (A \to B) \in \Gamma]\}$.

Define the *Routley star* operation $\cdot^*$: $\Gamma^* = \{A \mid \neg A \notin \Gamma\}$.

Let Th be the set of formal theories, Pr the set of prime formal theories.

The canonical model is the pre-model $M_C = (\mathsf{Th}, \mathsf{Pr}, \mathbf{B}, \circ, \subseteq, \cdot^*, \lambda\Gamma.(\Gamma \cap \mathsf{At}))$. The standard completeness proof proves that the canonical model is a model, and for each formal theory t, $t \models_{M_C} A$ iff $A \in t$.

2.3. **Explanation of the Semantic System.** We take roughly the view of Fine semantics advocated by Logan in [11]: the points of a Fine-style model represent *theories*, about whatever content is referred to by the symbols in At. A model is not a representation of a structure satisfying a particular theory (as with first-order logic models), but a representation of a way *the class of all theories* might be, or of a 'theory-building practice' or 'theory space' which lays out conditions under which a theory is proper *qua* theory. Thus the above model postulates specify the ways that *ways theories could be*, could be. These theories are not necessarily formal theories, they merely meet some vague notion of theory (see the 'bodies of information' in [12].)

$t \circ u$ represents the set of all conclusions that can be derived by using the 'derivation methods' endorsed by t on the assumptions in u. Thus the meaning of $t \models (A \to B)$ given by the definition of $\models$ is that one of these 'derivation methods' concludes B, given A.

ℓ is the theory whose 'derivation methods' are exactly those that all theories (in our theory space) are already closed under (therefore $\ell \circ t = t$), and we assume that any good theory-building practice will consider this a theory. $t \sqsubseteq u$ means that u is more informative than t. The verification relation $t \models A$ means that t is 'committed to' A. A *prime* theory p is intuitively one in which there is no purely disjunctive information - if $p \models (A \vee B)$, then $p \models A$ or $p \models B$. (From the definition of $\models$, this formal property is easily seen to be necessary for membership in P, although perhaps not sufficient.)

It is a principal difference between our style of models (Fine-style) and Routley-Meyer models that we allow points representing non-prime theories; in Routley-Meyer models all points are taken to be prime, and are often called 'worlds' rather than 'theories,' and $\models$ represents what is true at a world. (In a 'world', $A \vee B$ presumably can't be true unless one of the disjuncts is true.) Thus there is a philosophical difference (in spite of the mathematical connection) between the two kinds of semantics - if you believe that theory-building is the correct way to explain relevant logics, you might prefer Fine models, since non-prime theories are common in real life.

p^* is the theory which is committed to whatever is not excluded by p. (That this must be considered a 'theory' is perhaps the most questionable aspect of this system.) $v(t)$ is the set of atomic propositions to which t is committed.

In the canonical model, these vague notions are instantiated by precise notions applying to *formal* theories, and the Key Lemma ($t \models A$ iff $A \in t$) verifies that these precisifications agree with the intuition behind the semantics.

But why is all this an appropriate basis for a logic? Even relevant logicians do not all agree that it is. It is not our intent to survey all possible objections, but as one good example, we consider Brady's [10], where he argues that neither Fine-style nor Routley-Meyer-style models for relevant logics are well-motivated. We reconstruct (one of) his arguments as follows: the $\to$ connective should represent one of two things, truth-preservation or meaning-containment. If $\to$ is truth-preservation, it should follow classical laws. So if relevant logic is worth doing, $\to$ in relevant logics should express meaning-containment. But frame models provide too many possible semantics (given by different sets of optional postulates corresponding to axioms), and frame semantics itself provides no motivation for which of these kinds might describe meaning-containment.

We argue that the notion of (relevant) logic as the 'universal theory-building toolkit' is a third option, on which $A \to B$ is *not necessarily* a claim of truth-preservation *or* of meaning-containment. First, we presuppose that the meaning of a proposition such as $(A \to B)$ is always relative to a theory, and in the context of a fixed theory t, we say $A \to B$ is the claim that whenever we analyze some body of data u using the methods of t, and u contains A, we should conclude B is a consequence of u. These 'methods,' and the meaning of 'a consequence,' may vary depending on t - in particular, they could involve *either* preserving truth, or drawing out contained meanings. Logic is then thought of as a special theory, such that all theories worthy of the name are closed under *logical* consequence. So '$(A \to B)$ is a theorem of logic' means that every proper theory containing A contains B. What is a proper theory, and therefore what is in one's 'logic', depends on one's theory-building practice, which may have prejudices toward truth-preservation, meaning-containment, both, neither, or something else. But all these practices seem to have a pattern in common - they define a space of theories that looks like a Fine model. We are interested in those theorems that are in the logic of *every* theory-building practice. The canonical model shows that there is a 'weakest' theory-building practice whose internal logic is *just* this core of logic, which all the varieties of theory-builders can agree on. Thus our interest in non-canonical models of exactly **B** - these represent theory-building practices other than the 'weakest' one whose internal logic is just as general (weak). For example, if we could consider only *recursive* theories to be good theories, would this theory-building practice provide some distinctive logical principles that are not available to all theory-building practices? We will answer a question similar to this (though weaker) in what follows.

By nature, our view does not claim that logic *isn't* about truth-preservation, or about meaning-containment - sometimes. But since the activity of theory-building is more general than both, and turns out to have a non-trivial 'logic' of its own, we take this to be a conservative neutral foundation for logic.

Bear in mind that Brady's complaint that there are too many possible frame semantics for meaning-containment can also apply to theory-building; we do not have a conclusive argument that the model postulates corresponding to **B** are the right ones. However, there are good arguments to reject the postulates used by stronger logics such as **E** and **R** (see [11]); probably, if anything, the semantics should be weaker than the one presented here[2]. So the formal models we consider are perhaps at least a special case of the right models for theory-building.

3. Twin Points

Let M be a model. Let two points $t, u \in T_M$ be called *twin points* if, for all formulas A, $t \models A$ iff $u \models A$. The canonical model has no twin points, because any two different points are different

[2]For example, it might not validate distributivity; Brady's [10] argues for a distribution-free logic, and perhaps we should take theories built by use of this logic seriously as theories.

sets of formulas, and verify exactly the formulas they each contain. Now we construct a model of exactly the theorems of **B**, but containing twin points.

Preliminaries. Let M and M' be models and $\cdot' : T_M \to T_{M'}$ a function. $\cdot'$ is an *isomorphism* if it is bijective and:

(1) $(t \circ u)' = t' \circ u'$
(2) $t \in P_M$ iff $t' \in P_{M'}$
(3) $(p^*)' = p'^*$
(4) $t \sqsubseteq_M s$ iff $t' \sqsubseteq_{M'} s'$
(5) $v_M(t) = v_{M'}(t')$

Lemma 1. *Let M, M' be isomorphic models with isomorphism $\cdot'$, and $t \in M$. Then for all formulas A, $t \models_M A$ iff $t' \models_{M'} A$.*

Proof. By induction on the complexity of A.

Base case: A is atomic. Then $A \in v_M(t)$ iff $A \in v_{M'}(t')$ by definition.

Inductive cases:

$A = B \wedge C$: $t \models_M B \wedge C$ iff $t \models_M B$ and $t \models_M C$ iff $t' \models_{M'} B$ and $t' \models_{M'} C$, by IH, iff $t' \models_{M'} B \wedge C$.

$A = B \vee C$: Note that p is a prime extension of t in M iff p' is a prime extension of t' in M'. So $t \models B \vee C$ iff for all prime extensions p of t in M, $p \models_M B$ or $p \models_M C$, iff for all prime extensions p' of t' in M', $p' \models M'B$ or $p' \models_{M'} C$, iff $t' \models_{M'} B \vee C$.

$A = B \to C$: $t \models_M B \to C$ iff for all u in M such that $u \models_M B$, $t \circ_M u \models_M C$, iff for all $v \in M'$ such that $v \models B$, $t' \circ v \models_{M'} C$. This is because, by IH, these v are exactly the points which are u' for some $u \in M$, $u \models_M B$. Now let $t \circ u \models_M C$. Then $t' \circ v = t' \circ u' = (t \circ u)' \models_{M'} C$, by IH. Conversely, the $u \in M$ such that $u \models_M B$ are exactly the ones such that $u' \models_{M'} B$. Now let $t' \circ u' \models_{M'} C$. Then $(t \circ u)' \models_{M'} B$, so $t \circ u \models_M B$ by IH. This completes the iff, and the latter condition is $t \models_{M'} B \to C$.

$A = \neg B$: $t \models_M \neg B$ iff for all prime extensions p of t in M, $p^* \not\models_M B$, iff for all prime extensions p' of t' in M', $(p')^* \not\models_{M'} B$, (because $(p')^* = (p^*)'$, and we apply IH), iff $t \models_{M'} \neg B$. □

The Twin Points Construction. Let M be a model, and let M' be some model whose points are disjoint from T_M, and which is isomorphic to M. (Everyone knows two such models exist.) Let $\cdot^-$ denote the inverse of $\cdot'$. We define the model $N = (T_M \cup T_{M'}, P_M \cup P_{M'}, \ell_M, \sqsubseteq, \circ, *, v)$ where the last four components are defined as follows:

$t \sqsubseteq_N s$ iff $t, s \in T_M$ and $t \sqsubseteq_M s$, or $t, s \in T_{M'}$ and $t \sqsubseteq_{M'} s$
$t \circ s = t \circ_M s$ if both theories lie in M
$t \circ s = t \circ_{M'} s$ if both theories lie in M'
$t \circ s = t' \circ_{M'} s$ if $t \in M, s \in M'$
$t \circ s = t^- \circ_M s$ if $t \in M', s \in M$
(Note that an application $t \circ u$ lies in the same half of the model as u.)
$t^* = *_M(t)$ if $t \in M$, $*_{M'}(t)$ if $t \in M'$.[3]
$v(t) = v_M(t)$ if $t \in M$, $v_{M'}(t)$ otherwise.

It follows that $v(t) = v(t')$ for all $t \in M$.

[3]We do not write $*$ as a superscript here for readability, but it is the same function.

Note that the choice of ℓ is the only aspect of N that is not symmetrical.

Theorem 2. *N is a model.*

Proof. We must verify all the model postulates. First, $\sqsubseteq$ is a partial order: it is well known that a disjoint union of two partial orders is a partial order.

(1) Suppose $s \sqsubseteq t$. Then we know either s, t both lie in M or both in M'.
  (a) $v(s) \sqsubseteq v(t)$: if $s, t \in M$, true because M is a model. Otherwise, $s^- \sqsubseteq t^-$, so $v(s) = v_M(s^-) \sqsubseteq v_M(t^-) = v(t)$.
  (b) $u \circ s \sqsubseteq u \circ t$: Cases depending on (a) whether s, t lie in M, (b) whether u lies in M.
    11: because M is a model
    10: $u \circ s = u^- \circ_M s \sqsubseteq u^- \circ_M t = u \circ t$
    01: $u \circ s = u' \circ_{M'} s \sqsubseteq u' \circ_{M'} t = u \circ t$ (using the fact that M' is a model)
    00: because M' is a model
  (c) $s \circ u \sqsubseteq t \circ u$: Same case breakdown.
    11: because M is a model
    10: $s \circ u = s' \circ_{M'} u \sqsubseteq t' \circ_{M'} u = t \circ u$
    01: $s \circ u = s^- \circ_M u \sqsubseteq t^- \circ_M u = t \circ u$
    00: because M' is a model

(2) If $t \in M$, $\ell \circ t = t$. If $t \in M'$, $\ell \circ t = \ell' \circ_{M'} t = (\ell \circ_M t^-)' = (t^-)' = t$.

(3) If $p \in P_X$, where X is either M or M', $p^{**_N} = p^{**_X} = p$.

(4) if $p \sqsubseteq q$, then both p, q lie in M or both in M'. $q^* \sqsubseteq p^*$ holds because both M, M' are models.

(5) $v(t)$ is the intersection of $v(p)$ for all prime extensions of p: Note that if $t \in M$ the prime extensions of t (call this set $E(t)$) in N are exactly its prime extensions in M, because $t \sqsubseteq p$ iff $t \sqsubseteq_M p$, and $p \in M$ is prime in N iff it is prime in M. Therefore $v(t) = v_M(t) = \bigcap_{E(t)} v_M(p) = \bigcap_{E(t)} v(p)$. The same argument applies to M' points.

(6) Let $t \circ u \sqsubseteq p$, p prime. There are q, r extending t, u such that $q \circ u \sqsubseteq p$, $t \circ r \sqsubseteq p$:
  Note that either u, p are both in M or both in M'. So we have cases based on whether (a) $t \in M$ and (b) $u, p \in M$.
  11: because M is a model, and all prime points of M are still prime in N
  10: $t' \circ u = t \circ u \sqsubseteq p$, and we obtain suitable $q, r \in P_{M'}$ such that $q \circ u, t' \circ r \sqsubseteq p$. Since $t' \circ r = t \circ r$, $t \circ r \sqsubseteq p$. And q^- is a prime extension of t, and $q^- \circ u = q \circ u \sqsubseteq p$. So q^-, r are the required extensions.
  01: Similar, but get extensions q, r of t^-, u and use q', r.
  00: because M' is a model

□

Theorem 3. *For each point $t \in T_M$, and each formula A, $t \models_M A$ iff $t \models_N A$. For each point $t \in T_{M'}$, and each formula A, $t \models_{M'} A$ iff $t \models_N A$.*

Proof. By induction on the complexity of A. Within each step, without loss of generality, let $t \in T_M$. If $t \in M'$ the arguments are identical, with $\cdot'$ and $\cdot^-$ switched.

Base Case: A is an atom. Then by definition $v_N(t) = v_M(t)$, so $A \in v_N(t)$ iff $A \in v_M(t)$.

Inductive Cases:

$A = B \wedge C$: $t \models_N B \wedge C$ iff $t \models_N B$ and $t \models_N C$ iff $t \models_M B$ and $t \models_M C$ (by IH), iff $t \models_M B \wedge C$.

$A = B \vee C$: t is incomparable with all points of M' in $\sqsubseteq_N$, so its prime extensions in N are (by definition of primes of N) exactly the prime extensions of t in M, and by induction these points satisfy B (or C) in N exactly if they do so in N. So $t \models_N B \vee C$ iff for all prime extensions p of t in N, $p \models_N B$ or $p \models_N C$, iff for all prime extensions p of t in M, $p \models_M B$ or $p \models_M C$, iff $t \models_M B \vee C$.

$A = B \to C$: Let $t \models_N B \to C$, which holds iff for all $u \in N$, if $u \models_N B$, then $t \circ u \models_N C$. So if $u \in M$ and $u \models_M B$, by induction, $u \models_N B$, so $t \circ_N u \models_N C$. But $t \circ_N u = t \circ_M u$, therefore, by induction again, $t \circ_M u \models_M C$. So $t \models_M B \to C$.

Conversely, let $t \models_M B \to C$, and let $u \in N$ and $u \models_N B$. There are two cases. First, let $u \in M$. So $u \models_M B$ and $t \circ_M u = t \circ_N u \models C$, as required. Second, let $u \in M'$. So $u \models_{M'} B$. Then $t \circ_N u = t' \circ_{M'} u$. But since $\cdot'$ is an isomorphism, $t' \circ_{M'} u = (t \circ_M u^-)'$, and by lemma 1, $u^- \models_M B$, so $t \circ_M u^- \models_M C$, so by lemma 1 again, $(t \circ_M u^-) \models_{M'} C$. Then by inductive hypothesis, $t' \circ_{M'} u \models_N C$, as required.

$A = \neg B$: $t \models_N \neg B$ iff for all prime extensions p of t in N, $p^* \not\models_N B$, iff for all prime extensions p of t in M (which are the same) $p^* \not\models_M B$, iff $t \models_M \neg B$. □

Corollary 4. *N contains twin points.*

Proof. For any t in M, t, t' satisfy the same formulas in M, M' by lemma 1, so they are also twin points in N by the theorem above. □

3.1. **The Problem with Twin Points.** The problem with twin points in Fine models is that these models are supposed to represent the space of possible theories, and the content of a theory, if it is given by any data in the model, seems like it should be given by the set of formulas verified by the theory. So, how can twin points be theories and yet be distinct?

The above construction gives a hint: it creates a model which is really two models, and no theory of the one can be considered an 'extension' ($\sqsubseteq_N$) of a theory of the other. We might consider the atomic propositions to be meant entities, independent of language, and points of a theory to be concrete expressions of a body of propositions in language. The two submodels M, M' may contain the same information, but written in different languages. Think of extensions as containment of one set of written sentences in another. No theory of M' is an extension of a theory of M, if the theories in M are sets of English sentences and the theories in M' are sets of Spanish sentences. But we can still define an operator $t \circ u$ that works across languages, applying pure propositions to pure propositions and then writing down the result, so that $t \circ_N u$ is always written in the same language as u. But this is not a sensible way to define notions. Why is our application operator language-insensitive but our notion of extension is not? At any rate, we have proven that the conditions on Fine models are not so prescriptive that they exclude such bad notions, nor do these bad definitions alter the theorems of **B**.

Another perspective on twin points is that they allow us to model a notion of theories where theories sometimes contain strictly more information than is in their set of formulas. Perhaps they contain some fuzzy information, levels of importance, associations, emotional judgments, etc. which do not impact their logical consequences. But we would prefer not to introduce new formalism for such notions, and insist that every kind of theory content can in principle be translated into indicative propositions, using a larger vocabulary of symbols. This motivates a better construction of twin points:

The Symmetric Canonical Model. Let At' be a set strictly larger than At, and define formulas, models etc. over At' as above. Let M be a model with respect to At'. Now consider it as a pre-model wrt. At, by replacing the valuation function $v(t)$ with $v(t) \cap \mathsf{At}$. Call this pre-model the At-*reduct* of M.

Lemma 5. *Let M be the At-reduct of N. For each point $t \in M$, and $A \in \Phi_{\mathsf{At}}$, $t \models_M A$ iff $t \models_N A$*

Proof. By induction on the complexity of A.

Base case: If $A \in \mathsf{At}$, $A \in v_M(t)$ iff $A \in v_N(t) \cap \mathsf{At}$, iff $A \in v_M(t)$.

Inductive cases:

$A = B \wedge C$: $t \models_M B \wedge C$ iff $t \models_M B$ and $t \models_M C$ iff $t \models_N B$ and $t \models_N C$, iff $t \models_N B \wedge C$.

$A = B \vee C$: $t \models_M B \vee C$ iff for all prime extensions p of t in M, $p \models_M B$ or $p \models_M C$, but this is the same condition as for $t \models_N B \vee C$ because $\sqsubseteq_M = \sqsubseteq_N$ and $P_M = P_N$.

$A = B \to C$: $t \models_M B \to C$ iff for all $u \in M$ such that $u \models_M B$, $t \circ_M u \models_M C$. But the u in M are the u in N, and $B \in \Phi_{\mathsf{At}}$, so $u \models_M B$ iff $u \models_N B$ by IH, and $\circ_M = \circ_N$, so this is the same as the condition for $t \models_N B \to C$.

$A = \neg B$: $t \models_M \neg B$ iff for all prime extensions p of t in M, $p^* \not\models_M B$, which again is the same as the condition for $t \models_N \neg B$, because the $\cdot^*_M = \cdot^*_N$ and the prime extensions are the same in both models. □

Lemma 6. *Let M be a reduct of N, and let N be a model. Then M is a model.*

Proof. We must prove all the model postulates.

(1) Suppose $s \sqsubseteq t$ in M, and $u \in M$.
  (a) Then $v_N(s) \subseteq v_N(t)$ since N is a model, so $v_N(s) \cap \mathsf{At} \subseteq v_N(t) \cap \mathsf{At}$
  (b) $u \circ_M s = u \circ_N s \sqsubseteq u \circ_N t = u \circ_M t$. **Note** something about this condition: it is obviously true in M because it does not mention the valuation, the only component of M that differs from N. When this applies from now on, we will say 'trivial.'
  (c) trivial
(2) trivial
(3) trivial
(4) trivial
(5) $v_M(t) = \mathsf{At} \cap v_N(t) = \mathsf{At} \cap \bigcap_{t \sqsubseteq p \in P} v_N(p) = \bigcap_{t \sqsubseteq p \in P} (\mathsf{At} \cap v_N(p)) = \bigcap_{t \sqsubseteq p \in P} v_M(p)$
(6) trivial

□

Now let N be the canonical model with respect to At', and let M be its At-reduct. Then let σ be any permutation of At' leaving At fixed, and extend it to formulas and sets of formulas in the obvious way: $(A \wedge B)\sigma = (A\sigma \wedge B\sigma)$, $(A \to B)\sigma = (A\sigma \to B\sigma)$, $\Gamma\sigma = \{A\sigma \mid A \in \Gamma\}$, etc. This is clearly an injective function on sets of formulas, with inverse given by the extension of σ^-.

Lemma 7. *Let t be a formal theory over At'. $t\sigma$ is also a theory.*

Proof. Let $t\sigma \vdash A$. We will do induction on the length of derivation of the $\vdash$ judgment, to show $A \in t\sigma$.

Base case: $A \in t\sigma$. There is nothing to prove.

Inductive cases:

$A = B \wedge C, t\sigma \vdash B, t\sigma \vdash C$: Then B, C are in $t\sigma$, so there are $B', C' \in t$ such that $B = B'\sigma, C = C'\sigma$. Since t is a theory, $B' \wedge C' \in t$, and so $(B' \wedge C')\sigma = B \wedge C \in t\sigma$.

$(B \to A) \in \mathbf{B}$, $t\sigma \vdash B$: By the substitution property, $(B\sigma^- \to A\sigma^-)$ is a theorem of **B**. And by IH, $B \in t\sigma$, so $B\sigma^- \in t$. Thus since t is a theory, $A\sigma^- \in t$, so $A \in t\sigma$. □

Since the inverse of $t \mapsto t\sigma$ is $t \mapsto t\sigma^-$, σ is a permutation of the points of the canonical model. It is also an automorphism (an isomorphism from the canonical model to itself).

Lemma 8. *$t \mapsto t\sigma$ is an automorphism of M, the At-reduct of N.*

Proof. There are many conditions to prove.

(1) $(t \circ u)\sigma = \{B \mid (A \to B) \in t, A \in u)\sigma = \{B\sigma \mid (\exists A)(A \to B) \in t, A \in u\} = \{B\sigma \mid (\exists A\sigma)(A\sigma \to B\sigma) \in t\sigma, A\sigma \in u\sigma\} = t\sigma \circ u\sigma$

(2) Let p be prime. We must show $p\sigma$ is prime; the converse is immediate by applying the same argument to σ^-. So, if $A \vee B \in p\sigma$, $(A \vee B)\sigma^- \in p$, so $A\sigma^- \vee B\sigma^- \in p$, so either $A\sigma^- \in p$ or $B\sigma^- \in p$, so either $A \in p\sigma$ or $B \in p\sigma$.

(3) $(p^*)\sigma = \{B \mid \neg B \notin p\}\sigma = \{B\sigma \mid \neg B \notin p\} = \{B\sigma \mid \neg B\sigma \notin p\sigma\} = (p\sigma^*)$

(4) $t \subseteq s$ iff $t\sigma \subseteq s\sigma$ by basic property of images.

(5) $v(t\sigma) = t\sigma \cap \mathsf{At} = t \cap \mathsf{At}$, because σ leaves At fixed, so any atom A is in t iff it is in $t\sigma$, and this last expression is $v(t)$, so $v(t\sigma) = v(t)$ as required.

□

It now follows from lemma 8 and lemma 1 that M contains many twin points, because for every σ leaving At fixed and every t, t and $t\sigma$ satisfy the same formulas in M.

4. Missing Theories

Another way models may differ from the canonical model is to be less full. For every formal theory Γ, there is a point in the canonical model which verifies exactly the formulas in Γ, namely Γ itself. For a general point t in a model, let $\bar{t}$ be the set of formulas A such that $t \models A$. We will say a model M has a missing theory Γ if there is a formal theory Γ such that $\bar{t} \neq \Gamma$ for all $t \in T_M$.

Our one trick for constructing models with missing theories is to construct *countable* models, because a countable model necessarily has missing theories.

Justification of the Trick. For any set of formulas S, let $K(S)$ be the set of classical consequences of S.

Lemma 9. *$K(S)$ is a **B** formal theory.*

Proof. The rules of the B Hilbert system are classically admissible, and the axioms are classically valid. Therefore, all the theorems of **B** are classically valid. Now let $K(S) \vdash A$, and we prove by induction that $A \in K(S)$.

Base Case: $A \in K(S)$: there is nothing to prove.

Inductive Cases:

$A = B \wedge C$, $K(S) \vdash B$, $K(S) \vdash C$. Then by induction $B, C \in K(S)$, and in every classical valuation where B, C are both true, $B \wedge C$ is true. So every valuation satisfying S satisfies $B \wedge C$, so $B \wedge C \in K(S)$.

$(B \to A) \in \mathbf{B}$, $K(S) \vdash B$: So $\neg B \vee A$ is a classical validity, so if a valuation satisfies S, it satisfies B (by IH), and therefore also A. □

Lemma 10. *Let* At *be infinite. There are uncountably many formal theories.*

Proof. There are uncountably many sets of atoms S, and $K(S)$ satisfies exactly the atoms in S, because there is a classical valuation making exactly the atoms in S true. Therefore there are uncountably many distinct theories of the form $K(S)$. □

4.1. **Nonconstructive Existence.** First we will prove the existence of a countable model nonconstructively, using the fact that **B** models are essentially an elementary class. Define a first-order theory BMod as follows:

The language of BMod consists of one unary predicate $A(\cdot)$ for each atom $A \in \mathsf{At}$, a binary function symbol $\circ(\cdot,\cdot)$, a binary predicate symbol $*(\cdot,\cdot)$, a binary predicate $\sqsubseteq(\cdot,\cdot)$, a unary predicate $P(\cdot)$, and a constant symbol ℓ, and the special predicate $=$, with its usual semantics ($M, v \models (x = y)$ iff $v(x) = v(y)$).

BMod has the following axioms, where A is a schematic letter standing for all the predicates $A(\cdot)$ for $A \in \mathsf{At}$:

(1) $\forall x, y, z(\sqsubseteq(x, y) \wedge \sqsubseteq(y, z) \to \sqsubseteq(x, z))$
(2) $\forall x \sqsubseteq(x, x)$
(3) $\forall x, y(\sqsubseteq(x, y) \wedge \sqsubseteq(y, x) \to x = y)$
(4) $\forall x(P(x) \leftrightarrow \exists y(*(x, y)))$
(5) $\forall x, y, z(*(x, y) \wedge *(x, z) \to y = z)$

(1) (a) $\forall s, t(\sqsubseteq(s, t) \to (A(s) \to A(t)))$
  (b) $\forall s, t, u(\sqsubseteq(s, t) \to \sqsubseteq(\circ(u, s), \circ(u, t)))$
  (c) $\forall s, t, u(\sqsubseteq(s, t) \to \sqsubseteq(\circ(s, u), \circ(t, u)))$
(2) $\forall t(\circ(\ell, t) = t)$
(3) $\forall p, q, r(*(p, q) \wedge *(q, r) \to p = r)$
(4) $\forall p, q, p', q'(\sqsubseteq(p, q) \wedge *(p, p') \wedge *(q, q') \to \sqsubseteq(q', p')$
(5) $\forall t(A(t) \leftrightarrow \forall p(\sqsubseteq(t, p) \wedge P(p) \to A(p))$
(6) $\forall t, u, p(\sqsubseteq(\circ(t, u), p) \wedge P(p) \to \exists q, r(P(q) \wedge P(r) \wedge \sqsubseteq(t, q) \wedge \sqsubseteq(u, r) \wedge \sqsubseteq(\circ(t, r), p) \wedge \sqsubseteq(\circ(q, u), p)))$

Note that in any structure S satisfying the above axioms, $\sqsubseteq^S$ is a partial order, and $*^S$ is a function defined on the extent of P. For each point t in S, let $v(t) = \{A \in \mathsf{At} \mid A^S(t)\}$. The result is a pre-model we will call $P(S)$.

Theorem 11. *$P(S)$ is a **B** model iff S satisfies the axioms of* BMod.

Proof. By inspection, most of the axioms of BMod are logical translations of the conditions given in section 2.2, and the extra conditions that $\sqsubseteq$ is a partial order and $*$ is a function defined on P.

The only difficulty is with the axioms involving A, which we claim hold iff v satisfies the model conditions (1)(a) and (5).

$\forall s, t(\sqsubseteq (s,t) \to (A(s) \to A(t)))$ holds for all A iff, whenever $s \sqsubseteq t$ in $P(S)$, if $A \in v(s)$, $A \in v(t)$, which is just the meaning of $v(s) \subseteq v(t)$.

$\forall t(A(t) \leftrightarrow \forall p(\sqsubseteq (t,p) \wedge P(p) \to A(p))$ holds for all A iff, for every t, each A holds at t iff that A holds at all prime extensions of t, which means $A \in v(t)$ iff $A \in v(p)$ for all prime extensions p, which is just if $A \in \bigcap v(p)$. □

Now we extend BMod to track which theorems are true in a model:

Lemma 12. *For each formula A in the language of **B**, there is a first-order formula $\phi_A(t)$ with one free variable in the language of BMod such that for all models S of BMod and $t \in S$, $\phi^S_A(t)$ holds iff $t \models_{P(S)} A$.*

Proof. By induction on the complexity of A.

Base case: If A is an atom, the requisite formula is $A(t)$, because $A(t)$ holds in S iff $A \in v_{P(S)}(t)$ which holds iff $t \models_{P(S)} A$.

Inductive cases:

$A = B \wedge C$: $\phi_{B\wedge C}(t) = \phi_B(t) \wedge \phi_C t$. Now $\phi_A(t)$ holds iff $\phi_B(t), \phi_C(t)$ hold, iff $t \models B, t \models C$, iff $t \models A$.

$A = B \vee C$: $\phi_{B\vee C}(t) = \forall p(P(p) \wedge \sqsubseteq (t,p) \to \phi_B(p) \vee \phi_C(p))$. Now $\phi_A(t)$ holds iff, for all prime extensions p of t, either $\phi_B(p)$ or $\phi_C(p)$ holds, which by IH is equivalent to either $p \models B$ or $p \models C$ holding.

$A = B \to C$: $\phi_{B\to C} = \forall u(\phi_B(u) \to \phi_C(\circ(t,u)))$. Then $\phi_A(t)$ holds iff for all u, if $\phi_B(u)$ holds, then ϕ_C holds of $t \circ u$. But by IH that is equivalent to saying if $u \models B$ then $t \circ u \models C$, as required.

$A = \neg B$: $\phi_{\neg B} = \forall p, q(P(p) \wedge \sqsubseteq (t,p) \wedge *(p,q) \to \neg(\phi_B(q)))$. So $\phi_A(t)$ holds if for all p that are prime extensions of t and all q that are equal to p^*, ϕ_B does not hold at $q(= p^*)$. But by IH, this happens just if $p^* \not\models B$, as required. □

For any model M, we can define $A(\cdot)$ with extent $\{t \in T_M \mid A \in v(t)\}$, and the result is a structure S with $P(S) = M$, which by the above theorem is a model of BMod. So the canonical model demonstrates that there are models of BMod.

Theorem 13. *There exists a countable model M such that $M \models A$ iff $A \in$ **B**.*

Proof. Let M' be a model in which ℓ verifies exactly the theorems of **B**. As remarked, $M' = P(S)$ for some structure S satisfying BMod. Let $T(S)$ be the theory consisting of all sentences in the language of BMod satisfied by S. Since this language is countable, by the Lowenheim-Skolem theorem, there exists a countable substructure S' of S satisfying $T(S)$. So $P(S')$ is a countable model. Furthermore, $\ell_{S'} \models_{P(S')} A$ iff $\phi_A(\ell)$ holds in S', but this happens iff $\phi_A(\ell)$ holds in S, by the previous lemma, and since S' is a model of $T(S)$, this holds iff $A \in$ **B**. □

4.2. **Constructive Existence - Setup.** The Lowenheim-Skolem theorem is fine, but it doesn't tell us much about the model we've gotten. It has many missing theories, but which ones? We would like to get more explicit countable models. We will still do this by dropping points from the canonical model, but we will get a characterization of which theories are not missing.

More Lemmas. This material is mostly a standard part of the completeness proof for **B**.

Lemma 14. *All intersections of formal theories are formal theories.*

Proof. Let T be a set of theories, and take $\bigcap T$. Now let $\bigcap T \vdash A$, and we will prove by induction on the length of derivation that $A \in \bigcap T$.

Base case: $A \in \bigcap T$, and we're done.

Inductive cases:

$A = B \wedge C$, $\bigcap T \vdash B, \bigcap T \vdash C$. Then both B, C are in all theories t in T, and since these are theories, $B \wedge C$ is in all of them, so is in $\bigcap T$.

$(B \to A) \in \mathbf{B}$, $A \in \bigcap T$: Then $B \in t$ for all $t \in T$, so since these are theories, A is in all t, so $A \in \bigcap T$. □

Corollary 15. *For any set of formulas S, there is a smallest formal theory containing S.*

Proof. The trivial theory of all formulas contains S, so there is a nonempty set of theories containing S. Their intersection is then the unique smallest. □

Lemma 16. *$B \in \langle A \rangle$ iff $\{A\} \vdash B$.*

Proof. If $\{A\} \vdash B$, then any theory containing A also derives B, so B is in all such theories, by definition of a theory. Thus B is in the smallest such theory. For the converse, we prove $T = \{B \mid \{A\} \vdash B\}$ is a theory, which obviously contains A, from which it follows that this is $\langle A \rangle$, because its members are in all theories containing A.

By induction on the derivation $T \vdash B$.

Base Case: $B \in T$. There is nothing to prove.

Inductive Cases:

$B = C \wedge D$, $T \vdash C, T \vdash D$: Then $C, D \in T$ by IH, so $\{A\} \vdash C, \{A\} \vdash D$, so $\{A\} \vdash C \wedge D$, thus $C \wedge D \in T$.

$(C \to B) \in \mathbf{B}, T \vdash C$: Thus $C \in T$ by IH, so $\{A\} \vdash C$, so $\{A\} \vdash B$ since $(C \to B)$ is a theorem, so $B \in T$. □

Lemma 17. *Let $\bigwedge_i A_i$ represent $(A_1 \wedge (A_2 \wedge ...(A_n)))$, for some n indicated by context, and likewise $\bigvee_i A_i$. The following are theorems of **B**:*

(1) $(\neg A \wedge \neg B) \to \neg(A \vee B)$
(2) $\neg(A \wedge B) \to \neg A \vee \neg B$
(3) $(A \to B) \wedge (C \to D) \to (A \wedge C) \to (B \wedge D)$
(4) $\bigwedge_i (A_i \to B_i) \to (\bigvee_i A_i \to \bigvee_i B_i)$
(5) $\bigvee_i (A_i \to B_i) \to (\bigwedge_i A_i \to \bigvee_i B_i)$
(6) $\bigvee_i A_i \to A$ *when all* $A_i = A$

*The following are admissible rules in **B**:*

(1) $\dfrac{A \to B \qquad A \to C}{A \to B \wedge C}$
(2) $\dfrac{A \to B \qquad B \to C}{A \to C}$
(3) $\dfrac{A \to B}{\neg B \to \neg A}$

Proof. These can be derived from the Hilbert system for **B**. They can be found in the literature or used as exercises for the reader. □

Lemma 18. *$\{A\} \vdash B$ iff $A \to B$ is a theorem of **B**.*

Proof. The if direction is trivial. For the only-if direction, induction on the judgment $\{A\} \vdash B$.

Base case: $B \in \{A\}$. So $B = A$, and $A \to A$ is indeed a theorem.

Inductive cases:

$B = C \wedge D, \{A\} \vdash C, \{A\} \vdash D$. Then $A \to C, A \to D$ are theorems, and by lemma 17, $A \to (C \wedge D)$ is a theorem.

$(C \to B) \in$ **B**, $\{A\} \vdash C$: Then by IH, $A \to C$ is a theorem, so by lemma 17, so is $A \to B$. □

Lemma 19. *Let p be a prime formal theory. Then p^* is a prime theory.*

Proof. p^* is prime: Let $A \vee B \in p^*$. Then $\neg(A \vee B)$ is not in p. Thus $(\neg A \wedge \neg B)$ is not in p, since p is a theory, using lemma 17. So either $\neg A$ is not in p, or $\neg B$ is not in p, because if both were, their conjunction would be in p, since p is a theory. Thus either A or B is in p^*, by definition.

p^* is a theory: Let $p^* \vdash A$. We will prove $A \in p^*$ by induction on the judgment $p^* \vdash A$.

Base Case: $A \in p^*$: trivial.

Inductive Cases:

$A = B \wedge C$, $p^* \vdash B, p^* \vdash C$: Then by IH $B, C \in p^*$. So neither $\neg B$ nor $\neg C$ are in p. Thus since p is prime the disjunction $\neg B \vee \neg C$ cannot be in p, and since p is a theory $\neg(A \wedge B)$ is not in p (using a theorem from lemma 17). Thus $A \wedge B \in p^*$.

$(B \to A) \in$ **B**, $p^* \vdash B$: Then by IH $B \in p^*$, so $\neg B \notin p$. Now $(\neg A \to \neg B) \in$ **B**, using a rule from lemma 17, and p is a theory, so $\neg A$ cannot be in p either. Thus $A \in p^*$. □

Lemma 20. *Let t, u be formal theories. Then $t \circ u$ is a theory.*

Proof. Let $t \circ u \vdash A$. We do induction on this judgment.

Base Case: $A \in t \circ u$: trivial

Inductive Cases:

$A = B \wedge C$, $t \circ u \vdash B, t \circ u \vdash C$: Then $B, C \in t \circ u$ by IH, so there are $X, Y \in u$ such that $(X \to B), (Y \to C) \in t$. Thus $X \to B \wedge Y \to C \in t$, and it is a theorem of **B** that $(X \to B \wedge Y \to C) \to (X \wedge Y) \to (B \wedge C)$ (lemma 17); furthermore $X \wedge Y \in u$ since u is a theory. So $B \wedge C \in t \circ u$.

$(B \to A) \in$ **B**, $t \circ u \vdash B$: Then by IH $B \in t \circ u$, so there is some $X \in u$, $(X \to B) \in t$. $(X \to B) \to (X \to A)$ is a theorem of **B** (using a rule of the Hilbert system on $(B \to A)$, so $(X \to A) \in t$. Thus $A \in t \circ u$. □

Sub-Canonical Sets.

Definition 21. *Let a* sub-canonical set *of formal **B** theories be a set L such that:*

*(1) **B** $\in L$*
(2) For every formula A, $\langle A \rangle$, the smallest theory containing A, is in L
(3) For every $t \in L$, $t^* \in L$
(4) For every $t, u \in L$, $t \circ u \in L$
(5) For every $t, u \in L$ and every prime p in L such that $t \circ u \subseteq p$, there are prime extensions q, r of t, u such that $t \circ r \subseteq p$ and $q \circ u \subseteq p$.

(6) *For every $t \in L$, and A not in t, there is a prime extension p of t in L where $A \notin p$.*

Let L be a sub-canonical set, and define the pre-model $M_L = \langle L, P, \mathbf{B}, \subseteq, \circ, *, \lambda\Gamma.(\Gamma \cap \mathsf{At})\rangle$, where P is the set of prime theories in L, and $\circ, *$ are as defined for formal theories in section 2.2.

Theorem 22. *If L is a subcanonical set, M_L is a model.*

Proof. There are many model postulates to verify. Obviously $\subseteq$ is a partial order. M_L is closed under $*$ and $\circ$ because L is subcanonical.

(1) Suppose $s, t \in L$, $s \subseteq t$, and $u \in L$:
  (a) $v(s) = s \cap \mathsf{At} \subseteq t \cap \mathsf{At} = v(t)$
  (b) $u \circ s \subseteq u \circ t$: suppose $A \in u \circ s$. So there is $B \in s, (B \to A) \in u$. Then $B \in t$, so $A \in u \circ t$ for the same reason.
  (c) $s \circ u \subseteq t \circ u$: suppose $A \in s \circ u$. Then there is $B \in u$, $(B \to A) \in s$. Then $(B \to A) \in t$, so $A \in t \circ u$.

(2) $\mathbf{B} \circ t$ contains every $A \in t$, because $(A \to A) \in \mathbf{B}$. Conversely, if $B \in \mathbf{B} \circ t$, there is $(A \to B) \in \mathbf{B}$ and $A \in t$, and then because t is a theory, $B \in t$.

(3) $A \in p^{**}$ iff $\neg A \notin p^*$ iff $\neg\neg A \in p$, but since p is a theory and $A \to \neg\neg A$, $\neg\neg A \to A$ are theorems of $\mathbf{B}$, this is equivalent to $A \in p$.

(4) Let $p \subseteq q$ and $A \in q^*$. Then $\neg A \notin q$, so $\neg A \notin p$, so $A \in p^*$.

(5) Let $A \in \mathsf{At}$. If $A \in t$, A is in all prime extensions of t, so in their intersection. But if $A \notin t$, then since L is subcanonical there is a prime extension not containing A, so A is not in the intersection of all prime extensions.

(6) This property is in the definition of subcanonical set.

□

Theorem 23 (”Key Lemma”)**.** *If L is a subcanonical set, A is a formula, and $t \in M_L$, $t \models_{M_L} A$ iff $A \in t$.*

Proof. By induction on the complexity of A.

Base case: A is an atom. Then $t \models A$ iff $A \in v(t)$ iff $A \in t$.

Inductive cases:

$A = B \wedge C$: $t \models B \wedge C$ iff $t \models B$,$t \models C$, iff $B \in t$, $C \in t$, iff $B \wedge C \in t$ (because t is a theory).

$A = B \vee C$: $t \models B \vee C$ iff for all prime extensions p of t in L, $p \models B$ or $p \models C$, which for each p holds iff $B \in P$ or $C \in P$, and since p is a theory, p contains $B \vee C$ for all p. But since L is subcanonical, if $B \vee C \notin t$, it is not in some prime extension, so $B \vee C \in t$.

$A = B \to C$: Let $B \to C$ be in t. Then for all $u \models B$, by induction, $B \in u$, so $C \in t \circ u$. Therefore, by definition, $t \models B \to C$. Conversely, assume $t \models B \to C$. Consider $t \circ \langle B\rangle$. It must contain C. So there is some $X \to C$ in t such that $X \in \langle B\rangle$, which means by lemma 18 that $B \to X$ is a theorem of $\mathbf{B}$. But then $(X \to C) \to (B \to C)$ is a theorem of $\mathbf{B}$, so since t is a theory, $B \to C \in t$.

$A = \neg B$: $\neg B \in t$ iff for all prime extensions p of t, $\neg B \in p$, iff $B \notin p^*$, iff $p^* \not\models B$ (by IH), iff $t \models \neg B$ (by definition). □

5. Constructive Existence - Pseudo-Recursion-Theoretic Construction

Theorems 22 and 23 show that if we can find a countable subcanonical set of theories, we will have a countable model which verifies exactly the theorems of **B**, which model necessarily has many missing theories. A subcanonical set is closed under several theory-building operations, such as applications $t \circ u$, complements p^*, prime extensions avoiding certain formulas, etc. Theories described computationally are natural candidates, if these constructions derive theories that are still 'computational'. To that end, we prepare a recursive encoding of our logic. Assume the following facts about recursive functions on natural numbers:

(1) There exists a constant $\emptyset$ and recursive *list-forming function* $\mathsf{append}(l, n)$, *projection function* $p(l, i)$ such that (letting $\mathsf{append}(x_1, ...x_n)$ be an abbreviation for $\mathsf{append}(\mathsf{append}(...(\emptyset, x_1), ...), x_n)$:
  - (a) $p(\mathsf{append}(x_1, ...x_n), i) = x_i$
  - (b) there exists a *length function* len such that $\mathsf{len}(\mathsf{append}(x_1, ...x_n)) = n$
  - (c) for any recursive predicate $P(\cdot)$, there is a recursive predicate accepting exactly numbers of the form $\mathsf{append}(x_1, ...x_n)$ where $P(x_1), ...P(x_n)$ hold

(2) There exists an injective mapping enc from formulas over At to natural numbers, satisfying these properties:
  - (a) There is a recursive function $f_\wedge$ such that $\mathsf{enc}(A \wedge B) = f(\mathsf{enc}(A), \mathsf{enc}(B))$, and likewise for $\neg, \vee, \rightarrow$.
  - (b) $\mathsf{enc}[\Phi_{\mathsf{At}}]$ is recursive. (Call the function that tests for this property i_Φ.)
  - (c) There is a recursive function $p_{\wedge,1}$ such that $p_{\wedge,1}(\mathsf{enc}(A \wedge B)) = \mathsf{enc}(A)$, and likewise for position 2 and the other connectives.

(3) For any finite Hilbert system, there is a recursive predicate $P(p, a)$ which holds exactly when p has the form $\mathsf{append}(\mathsf{enc}(A_1), ...\mathsf{enc}(A_n))$ and $a = \mathsf{enc}(A_n)$, for A_n a formula and $A_1, ...A_n$ some Hilbert proof of A_n. (A finite Hilbert system is a Hilbert system in which there are a finite set of axioms and rules of which all axioms and rules are substitution instances.)

Why Not Recursive Theories? The most obvious choice of computational theories, the recursive theories, are not a subcanonical set - if t, u are recursive, the most we can say about $t \circ u$ is that it is recursively enumerable. We leave as an open problem whether there are submodels of the canonical model satisfying exactly **B** in which all points are recursive theories.

Let a set of formulas S be called recursive if $\mathsf{enc}[S]$ is recursive.

Lemma 24. *Let s, t be* classical *theories (i.e. sets closed under classical entailment), and s be nonempty. Then $t \circ s$ is the smallest classical theory containing both t and s.*

Proof. $t \circ s$ contains s because for every A, $A \rightarrow A$ is in t, as this is in every classical theory. $t \circ s$ contains t, because if $(A \rightarrow B)$ is in t, so is $C \rightarrow (A \rightarrow B)$ for any C; choose a C that lies in s, and we get $(A \rightarrow B) \in t \circ s$. $t \circ s$ is a classical theory: let C be a consequence of $t \circ s$. Then by compactness C is a consequence of $\bigwedge B_i$ for some finitely many B_i, such that $A_i \in s$, $(A_i \rightarrow B_i) \in t$. Then $(\bigwedge A_i \rightarrow \bigwedge B_i) \in t$, and therefore $(\bigwedge A_i \rightarrow C) \in t$, and $\bigwedge A_i \in s$, thus $\bigwedge C \in t \circ s$.

$t \circ s$ is the smallest theory containing t, s: let r be a theory containing t, s. Then for every $B \in t \circ s$, it contains some $(A \rightarrow B)$ and A. But B is a consequence of these, so $B \in r$. Thus $t \circ s \subseteq r$. ◻

Lemma 25. *Let t', s' be the smallest* classical *theories containing t, s, where t is a set of formulas of the form $(A \to B)$ where A, B are atoms, and s is a nonempty set of atoms. Further suppose no atom appears in both an antecedent and a consequent in t. Then the atoms in $t' \circ s'$ are those in s and in $t \circ s$.*

Proof. Obviously $t' \circ s'$ contains all these atoms. Let C be any atom not in s or $t \circ s$. Now consider the truth function v making true exactly the atoms in $s \cup (t \circ s)$. v satisfies s, and also satisfies t: let $(A \to B)$ be in t. Since A cannot be a consequent of any conditional in t, $v(A) = 1$ iff $A \in S$. Thus if A is true in v, $B \in t \circ s$ and therefore true, and if A is false, $(A \to B)$ is vacuously true. Therefore v is a model of both theories t', s', so by lemma 24, the theory $T(v)$ of formulas true in v is an extension of $t' \circ s'$, and does not contain C, so $C \notin t' \circ s'$. □

Theorem 26. *There exist recursive formal theories t, s such that $t \circ s$ is not recursive.*

Proof. For each Turing machine (code) M and each integer i, assume we have unique proposition letters $\langle M, i\rangle$ and $\langle M\rangle$, and a recursive (both ways) bijection from $(M, i) \mapsto \mathsf{enc}(\langle M, i\rangle)$ and likewise for $M \mapsto \langle M\rangle$. Now let s be the classical theory generated by all $\langle M, i\rangle$, and t that generated by all $(\langle M, i\rangle \to \langle M\rangle)$ such that M halts on an empty tape within i steps. These are, in particular, **B** formal theories. Both of these theories are clearly recursive. But by the previous lemma, the atoms in $t \circ s$ are exactly the $\langle M\rangle$ such that M halts in some number of steps, and therefore the halting problem reduces to membership in $t \circ s$. □

Neutral Fixed-Point Logic. Now we will introduce a logic interpreted over first-order structures which we will call *neutral fixed-point logic*. This logic uses *predicate expressions*, each with an associated arity, which may be primitive predicate symbols, predicate variables, or expressions of the form $(\iota P)(\phi, \vec{x})$. Here P is a predicate variable of arity $|\vec{x}|$ and ϕ is a formula. The arity of this expression is considered to be the same as the arity of P. Formulas are as follows:

(1) $P(\vec{x})$ where $\vec{x}$ are variables and P is a predicate expression of arity $|x|$
(2) $\phi \wedge \psi$
(3) $\neg\phi$
(4) $\forall x \phi$

Use the notation $v[x \backslash a]$ for the function which maps every variable $y \neq x$ to $v(y)$, and x to a.

Relative to a structure M and a variable assignment v (assigning both individual and predicate variables), we recursively define (three-valued) truth of formulas and meaning of predicate expressions. Formulas can be true, false, or meaningless. Predicate expressions are either meaningless in M or have an extent in M, which is a relation on $|M|$. $M, v \models \phi$ means 'ϕ is true in M with assignment v'.

(1) $P(\vec{x})$ is true for primitive P, or meaningful expressions P, if $v[\vec{x}] \in P^M$, otherwise false
(2) $P(\vec{x})$ is true for variable P if $v[\vec{x}] \in v(P)$, otherwise false
(3) $P(\vec{x})$ is meaningless if P is a meaningless expression
(4) $x = y$ is true iff $v(x) = v(y)$
(5) $\phi \wedge \psi$ is true if both ϕ and ψ are true, meaningless if either ϕ or ψ is meaningless, otherwise false
(6) $\neg\phi$ is false if ϕ is true, true if ϕ is false, and meaningless if ϕ is meaningless

(7) $\forall x\phi$ is true if, for all variable assignments $v' = v[x\backslash a]$ for $a \in |M|$, ϕ is true with respect to M, v', meaningless if ϕ is meaningless with respect to any of the M, v', and otherwise false

(8) $(\iota P)(\phi, \vec{x})$: If R_P is a relation of arity $|\vec{x}|$ such that for all variable assignments v where $v(P) = R_P$, $M, v \models \phi$ iff $M, v \models P(\vec{x})$, call R_P a *solution* to the iota-expression. If there is a unique solution R_P, then $(\iota P)(\phi, \vec{x})^M = R_P$. Otherwise the expression is meaningless.

Commentary. Neutral fixed-point logic, like more common fixed-point logics (see [13] for reference), allows formulas to act as operators deriving relations from relations, and allows definition of relations that are fixed points of these operators - intuitively, recursive definitions. Unlike least fixed-point logic, we allow negative occurrences of the relation being defined in its inductive definition. but we also do not do anything to guarantee existence, as in inflationary fixed-point logic (which uses a different notion of 'fixed-point' than the usual one). So for some syntactically-valid definitions there is no fixed point. Partial fixed-point logic solves the same problem by declaring that all such definitions define the empty relation. But we do not use such a default - for us, constructs with no fixed point are just bad definitions, and any formula in which they appear takes no truth value. Furthermore, these logics have ways to deal with multiple fixed points (e.g. take the least). We do not. In neutral fixed-point logic, a definition that defines ambiguously is invalid, and formulas containing it have no truth value.

Say that a relation R over $|M|$ is *definable* (in neutral fixed-point logic) if there is a formula ϕ with free variables $\vec{x}$ such that for every variable assignment v, we have $M, v \models \phi$ iff $v[\vec{x}] \in R$. *Our explicit subcanonical set of theories will be the theories t such that $\mathsf{enc}(t)$ is definable in $\langle \mathbb{N}, S \rangle$, where S is the successor function.* We will call these theories *definable theories*.

A standard lemma about nesting definitions:

Lemma 27. *If relations $R_1, ...R_m$ are definable in M, and a relation S is definable in the extension of the structure M with additional predicate symbols P_i such that $P_i^M = R_i$, then S is definable in M.*

Proof. Let $\phi, \vec{x}$ be the formula and vector of variables that define S. For each R_i there is a formula ψ_i and variables $\vec{y}_i$ which define it in M. Then let ϕ' be the formula over the signature of M obtained by replacing every instance of $P_i(\vec{z})$ in ϕ with $\psi_i[\vec{y}_i\backslash\vec{z}]$. We claim $\phi', \vec{x}$ defines S in M.

From the definition of defining, and induction on the structure of ϕ, we see that $M, v \models \phi'$ iff $M, v \models \phi$, which happens iff $v[\vec{x}] \in S$, so ϕ' defines S. □

So when a relation is definable in our fixed structure $\langle \mathbb{N}, S \rangle$, we will freely use it as if it were a predicate symbol from now on.

Lemma 28. *All recursive functions (treated as relations) are definable in $\langle \mathbb{N}, S \rangle$.*

Proof. We prove this by induction on the depth of definition of a recursive function. There is one case for each of the six standard rules that generates recursive functions. We will use the convention that a function $f(\vec{x})$ is represented by a relation $f(\vec{x}, z)$ which holds iff $f(\vec{x}) = z$; we write "$f(\vec{x} = z$" for this relation.

(1) The constant function $f(n) = a$: Since $\{0\}$ is definable by $\forall m\neg(S(m, n))$, by induction $\{a\}$ is definable for every natural number a. Thus $f : n \mapsto a$ is definable: $f(x) = z$ holds iff $(x = x) \wedge (\{a\}(z))$.

(2) The successor function is trivially definable because its corresponding relation is in the signature of $\langle \mathbb{N}, S \rangle$.
(3) The projection function $\mathsf{proj}_{n,i} : (x_1, ...x_n) \mapsto x_i$: the relation $\mathsf{proj}_{n,i}(\vec{x}) = z$ is defined by $[(x_1 = x_1) \wedge ...(x_n = x_n)] \wedge (z = x_i)$.
(4) If f and $g_1, ...g_m$ are definable, their composition $f(g_1(\vec{x}), ...g_m(\vec{x})) = z$ is defined by $\exists z_1, ...z_m[f(z_1, ...z_m, z) \wedge g_1(x_1, ...x_n, z_1) \wedge ...g_m(x_1, ..x_n, z_m)]$, with free variables $\langle x_1, ...x_n, z\rangle$.
(5) If g, h are definable, the relation $f(n, \vec{x}) = z$ where f is defined by for-recursion with base case g and recursion operation h, and n is the variable of recursion, is the extent of the expression $(\iota F)(\phi, (n, x_1, ...x_k, z))$, where $\phi = [((n = 0) \wedge g(x_1, ...x_k) = z) \vee (S(m, n) \wedge \exists z'[F(m, x_1, ...x_k) = z' \wedge h(z', x_1, ...x_k) = z])]$. Let $F'(...) = ...$ be any relation with arity $|n, x_1, ..x_k, z|$ such that for all variable assignments v mapping the symbol $F(...) = ...$ to $F'(...) = ...$, $\langle \mathbb{N}, S \rangle, v \models \phi$ iff $F'(v(n), v(\vec{x})) = v(z)$ holds. It is easy to see that f itself is such a relation, but we need uniqueness. We prove by induction on $v(n)$ that $F'(v(n), v(\vec{x})) = v(z)$ holds iff $f(v(n), v(\vec{x})) = v(z)$ holds (the latter = sign is not just notation!) Base case: $v(n) = 0$, F'... holds iff $g(v(\vec{x})) = v(z)$, iff $f(0, v(\vec{x})) = v(z)$. Inductive case: F' holds iff for some z', $h(z', v(\vec{x})) = v(z)$, where $F'(v(n) - 1, v(\vec{x})) = z'$. By induction, z' is the unique value of $f(v(n) - 1, v(\vec{x}))$. So $v(z) = h(f(v(n) - 1, v(\vec{x})), v(\vec{x})) = f(v(n), v(\vec{x}))$, as required.
(6) If $f(n, \vec{x})$ is definable, so is $(\mu n) f(n, \vec{x})$, which is a function of only $\vec{x}$. First, note that the $<$ relation is definable, because its indicator function, being primitive recursive, is definable by the above cases. $[(\mu n) f(n, \vec{x})](\vec{x}) = z$ is defined by $f(z, \vec{x}) = 0 \wedge \forall n((n < z) \to \exists z'(f(n, \vec{x}) = z' \wedge z' > 0))$.

□

Lemmas about Definable Theories. Let a *derivation sequence* for B from a set of formulas X with *source* $S \subseteq X$ be a sequence of formulas $A_0, ...A_n$ where $A_0 = A$, $A_n = B$, such that for each i, one of the following:

(1) $A_i \in S$
(2) $A_i = A_j \wedge A_k$ for $j, k < i$
(3) $(A_j \to A_i) \in \mathbf{B}$ for some $j < i$

Lemma 29. *$X \vdash B$ iff there is a derivation sequence from X to B with some finite source $S \subseteq X$.*

Proof. First, let $X \vdash B$. We construct a sequence by induction on the proof judgment.

Base case: $B \in X$: Then the sequence $\langle B\rangle$ will do, with source $\{B\}$.

Inductive cases:

$B = C \wedge D$, $X \vdash C$, $X \vdash D$: Then there are derivation sequences $A_0, ...A_n$ and $B_0, ...B_n$ from A to C and A to D, with sources S_C, S_D. Then $A_0, ...A_n, B_0, ...B_n, (C \wedge D)$ is a derivation sequence, because $A_n = C$, $B_n = D$, with source $S_C \cup S_D$.

$A \vdash C$, $(C \to B) \in \mathbf{B}$: Then there is by IH some derivation sequence $A_0, ...A_n$ from A to C with source S, and $A_0, ...A_n, B$ is a derivation sequence for B with source S because $A_n = C$.

For the converse, the finiteness of the source doesn't matter. We will do induction on the length of the derivation sequence.

Base case: length 1. So $A_0 = A_n = A = B$, so $B \in S$, so $X \vdash B$.

Inductive cases:

$A_n = A_i \wedge A_j$. So, $A_0, ...A_i$ and $A_0, ...A_j$ are shorter sequences, so by IH, $X \vdash A_i$, $X \vdash A_j$. Thus $X \vdash A_i \wedge A_j$.

$(A_i \to A_n) \in \mathbf{B}$: Then $A_0, ...A_i$ is a shorter sequence, so by IH, $X \vdash A_i$, so $X \vdash A_n$. □

Lemma 30. *The set of encodings of theorems of **B** is definable.*

Proof. By assumption, there is a recursive predicate $P(p, a)$ which holds exactly when $p = \mathsf{append}(\mathsf{enc}(A_1), ...\mathsf{enc}(A_n))$ and $a = \mathsf{enc}(A_n)$, for A_n a formula and $A_1, ...A_n$ some Hilbert proof of A_n. But since the theorems of **B** are just the formulas for which such a Hilbert proof exists, the formula $\exists p[P(p, a)]$ with free variable a defines the theorems of **B**. □

Lemma 31. *There is a definable predicate Der such that $\mathsf{Der}(p, s, a)$ holds iff $a = \mathsf{enc}(A)$ for some formula A, and p is the encoding of a derivation sequence for A with source S, and s is a list containing the encoded members of S.*

Proof. By assumption, there is a function i_Φ which checks whether its input is an encoded formula. So there is also a recursive function i^*_Φ which checks whether its input is a list of encoded formulas. We will also use the recursive predicate $\in(\cdot, \cdot)$ which checks whether a given number is in a given list. Let **B** denote the defined predicate for the set of encoded theorems of **B**. Let $\chi_\to$ be the predicate which checks whether a number is an encoding of a formula of the form $A \to B$. This can be defined by $(i_\Phi(p_{\to,1}(n)) = 1) \wedge (i_\Phi(p_{\to,2}(n)) = 1) \wedge (n = f_\to(p_{\to,1}(n), p_{\to,2}(n)))$.

$\mathsf{Der}(p, s, a)$ is now defined by:

$(i^*_\Phi(p) = 1) \wedge (i^*_\Phi(s) = 1) \wedge (p[\mathsf{len}(p)] = a) \wedge \forall i[i < \mathsf{len}(p) \to (\in(p[i], s) \vee \exists j, k(j < i \wedge k < i \wedge p[i] = f_\wedge(p[j], p[k])) \vee \exists T, j(\mathbf{B}(T) \wedge \chi_\to(T) \wedge j < i \wedge p[j] = p_{\to,1}(T) \wedge p[i] = p_{\to,2}(T)))]$ □

Mutually Recursive Defining Formulas. It will be convenient for the next proof to extend neutral fixed-point logic with a further type of predicate expression, $(\iota P_i \in P_1, ...P_n)(\phi_1, \vec{x}_1, ...\phi_n, \vec{x}_n)$, such that the vectors of variables $\vec{x}_1, ...\vec{x}_n$ share no variables. Let a tuple of relations R_i such that for all variable assignments v that send each P_i to R_i, $M, v \models \phi_i$ iff $v[\vec{x}_i] \in R_i^M$ be called a solution. The expression is meaningful iff it has a unique solution $R_1, ...R_n$, and we will say the relations $R_1, ...R_n$ are semi-defined by the expression. When the expression $(\iota P_k \in P_1, ...P_n)(\phi_1, \vec{x}_1, ...\phi_n, \vec{x}_n)$ is meaningful, its extent in M is R_k. Call this extended logic *multiary* neutral fixed-point logic.

This does not increase our defining power, as we now show.

Lemma 32. *Let M be a structure that can define at least one nontrivial unary predicate P, i.e. P^M and $(\neg P)^M$ are both nonempty. Every relation definable over M in multiary neutral fixed-point logic is definable over M in neutral fixed-point logic.*

Proof. It suffices to show that the extent of a single $(\iota P_i \in P_1, ...P_n)(\phi_1, \vec{x}_1, ...\phi_n, \vec{x}_n)$ expression can be defined.

Given any tuple of relations R_i, consider the relation R_Q of arity $(\sum_i \mathsf{arity}(R_i)) + n$ such that $R_Q(\vec{x}_1, ...\vec{x}_n, p_1, ...p_n)$ holds iff for some $i \in [1...n]$, $p_i \in P^M$, and for all $j \neq i \in [1...n]$, $p_j \notin P^M$, and $R_i(\vec{x}_i)$ holds. If R_Q is definable, each R_i is definable, by $\exists((\vec{x}_1, ...\vec{x}_n) \setminus \vec{x}_i), p_1, ..p_m[P(p_i) \wedge \bigwedge_{j \neq i} \neg P(p_j) \wedge R_Q(\vec{x}_1, ...\vec{x}_n)]$, with free variables $\vec{x}_i$.

Given an expression E_m of the form $(\iota P_i \in P_1, ...P_n)(\phi_1, \vec{x}_1, ...\phi_n, \vec{x}_n)$, which semi-defines $R_1, ...R_n$ (so its extent is R_i), we will show how to define R_Q. Note that the vectors of variables $\vec{x}_1, ...\vec{x}_n$

are disjoint. Let Q be a predicate variable of arity $(\sum_i \mathsf{arity}(P_i)) + n$, and let E_i be the formula $\exists((\vec{x}_1, ...\vec{x}_n) \setminus \vec{x}_i), p_1, ..p_m[P(p_i) \wedge \bigwedge_{j \neq i} \neg P(p_j) \wedge Q(\vec{x}_1, ...\vec{x}_n)]$. Let ψ_i^* be ϕ_i with each instance of $P_j(\vec{y})$, for any $j, \vec{y}$, replaced with $E_i[\vec{x}_j \backslash \vec{y}]$, and ψ_i be $\psi_i^* \wedge P(p_i) \wedge \bigwedge_{j \neq i} \neg P(p_j)$.

We claim R_Q is the extent of the expression
$E_Q = (\iota Q)(\psi_1 \vee ... \vee \psi_n, (\vec{x}_1...\vec{x}_n, p_1, ...p_n))$.

First, we prove that R_Q is a solution to E_Q. Let v be any assignment sending Q to R_Q. We must prove that $v[\vec{x}_1...\vec{x}_n, p_1, ...p_n] \in R_Q$ iff $M, v \models \psi_1 \vee ...\psi_n$. Let v' be the variable assignment that agrees with v except that it sends P_i to R_i for all i.

$v[\vec{x}_1...\vec{x}_n, p_1, ...p_n] \in R_Q$ iff
for exactly one i, $v(p_i) \in P^M$, and $v[\vec{x}_i] \in R_i$, which holds iff
for exactly one i, $v'(p_i) \in P^M$ and $v'[\vec{x}_i] \in R_i$, which holds iff
for exactly one i, $v'(p_i) \in P^M$ and $M, v' \models \phi_i$ (because E_m semi-defines R_i), which holds iff
for exactly one i, $v'(p_i) \in P^M$ and $M, v' \models \psi_i^*$ (because $v(Q) = R_Q$, and E_j "defines" R_j if Q is fixed to R_Q), which holds iff
for exactly one i, $M, v' \models \psi_i$, which holds iff
for exactly one i, $M, v \models \psi_i$ (because P_i do not occur in this formula), which holds iff
$M, v \models \psi_1 \vee ...\psi_n$

Next, we need to prove R_Q is the unique solution to E_Q. So let R'_Q be any solution.

Let the relations R'_i be those defined by the formulas E_i when Q is fixed to R'_Q, that is, $R'_i(\vec{x}_i)$ holds iff for some $\vec{x}_1...\vec{x}_n, p_1, ...p_n, R'_Q(\vec{x}_1...\vec{x}_n, p_1, ...p_n)$ holds. Then for every variable assignment v mapping all the P_i to R'_i, there is a variable assignment $v' = v[Q\backslash R'_Q]$ such that $M, v \models \phi_i$ iff $M, v' \models \psi_i^*$, since the E_i define R'_i when Q is fixed to R'_Q. Furthermore, for any i there is a variable assignment v''_i equal to v' except on the variables p_j, such that $v''_i(p_i) \in P^M$ and $v''_i(p_j) \notin P^M$ for $j \neq i$, and $M, v''_i \models \psi_i$ iff $M, v' \models \psi_i^*$ (because p_j do not occur free in ϕ_i or ψ_i^*). Now for every v sending all P_j to R'_j, $M, v \models \phi_i$ for some i iff $M, v' \models \psi_i^*$ for some i, iff $M, v''_i \models \psi_i$ for some i, iff $v''_i[\vec{x}_1...\vec{x}_n, p_1, ...p_n] \in R'_Q$ for some i, iff $v''_i[\vec{x}_i] \in R'_i$ for some i, iff $v[\vec{x}_i] \in R'_i$, which means R'_i must be R_i for all i because E_m semi-defines R_i.

Therefore, let v be any variable assignment sending Q to R'_Q, and now let v' be $v[P_i\backslash R_i]$.

$v[\vec{x}_1...\vec{x}_n, p_1, ...p_n] \in R'_Q$ iff
for exactly one i, $M, v \models \psi_i$ iff
for one i, $v(p_i) \in P^M$ and $M, v \models \psi_i^*$ iff
for one i, $v(p_i) \in P^M$ and $M, v' \models \phi_i$ iff
for one i, $v(p_i) \in P^M$ and $v'[\vec{x}_i] \in R_i$ (because E_m semi-defines R_i) iff
for one i, $v(p_i) \in P^M$ and $v[\vec{x}_i] \in R_i$,
which shows that R'_Q meets the definition of R_Q.

So, R_Q is the unique solution, and E_Q defines R_Q. Thus we can define each R_i, as argued above. □

Lemma 33. *There is a definable predicate $P(n, a)$ which holds iff $a = \mathsf{enc}(f(n))$ where $f : \mathbb{N} \to \Phi_{At}$ is an enumeration of all formulas with main connective $\vee$.*

Proof. As above, we can define the set of formulas with main connective $\vee$, using the $f_\vee$ and $p_\vee, i$. Call this set O. Then $P(n, a)$ can be defined by saying that $P(0, a)$ holds iff a is the least encoding of an $\vee$-formula, and $P(n, a)$ holds iff a has not yet been enumerated, but all smaller $\vee$-formulas have. That is,

$((n = 0) \wedge O(a) \wedge \forall b[b < a \rightarrow \neg O(b)]) \vee ((n > 0) \wedge O(a) \wedge \forall b[(b < a \wedge O(b)) \rightarrow \exists m[(m < n) \wedge P(m, b)]] \wedge \forall m[m < n \rightarrow \neg P(m, a)])$ □

Lemma 34 (Strengthened Lindenbaum Lemma)**.** *Let t be a definable theory, and let* Δ *be a definable set of formulas disjoint from t and closed under disjunctions. There exists a definable prime extension p of t, which is disjoint from* Δ*.*

Proof. This proof contains the standard construction for the Lindenbaum extension in relevant logic settings, and we show that the extension obtained this way is indeed definable.

By lemma 33, the formulas with main connective $\vee$ are enumerated, $L_1 \vee R_1, L_2 \vee R_2, L_3 \vee R_3, ...,$ such that the relation '*a* is the encoding of the *n*-th $\vee$-formula' is definable.

Define a family of sets t^i_j inductively as follows:

- $t^0_0 = t$
- $t^i_{j+1} = t^i_j$ if $t^i_j \nvdash L_j \vee R_j$; otherwise, if $\langle t^i_j \cup \{L_j\}\rangle$ is disjoint from Δ, $t^i_{j+1} = t^i_j \cup \{L_j\}$, otherwise $t^i_{j+1} = t^i_j \cup \{R_j\}$.
- $t^{i+1}_0 = \bigcup_j t^i_j$

Let $p = \bigcup_i t^i_0$. We claim p is the desired prime extension.

First, p is a theory. Suppose that $p \vdash A$. Then $p \vdash A \vee A$, by an axiom of **B**, and there is a finite subset of p which derives $A \vee A$, so for some i, $t^i_0 \vdash A \vee A$. There is also k such that $A \vee A = L_k \vee R_k$. Thus, $t^i_k \vdash A \vee A$, so either L_k or R_k is in t^i_{k+1}, but both of those are A, so $A \in t^i_{k+1} \subseteq t^{i+1}_0 \subseteq p$. Thus p contains A, as needed.

Second, p is prime. Suppose $A \vee B \in p$. Then $A \vee B \in t^i_0$ for some i, and $A \vee B = L_k \vee R_k$ for some k. So $A \vee B \in t^i_k$, so $t^i_k \vdash L_k \vee R_k$, so by definition t^i_{k+1} contains either L_k or R_k, that is, A or B. So either A or B is also in the larger set p.

Third, p is disjoint from Δ. Suppose p and Δ are not disjoint. Then p derives some formula in Δ. Then t^i_0 derives a formula in Δ for some i, and so does $t^{i'}_0$ for all $i' > i$. So let i be the largest i such that t^i_0 derives nothing in Δ. There is such a largest, because t^0_0 is disjoint from Δ by assumption and is a theory. Then since t^{i+1}_0 derives something in Δ, some t^i_j does as well. Again let j be the largest j such that t^i_j does not derive any formula in Δ. So there is $A \in \Delta$ such that $t^i_{j+1} \vdash A$. So $t^i_{j+1} \neq t^i_j$. So, by definition, it must be that $t^i_j \vdash L_j \vee R_j$, and either: $\langle t^i_j \cup \{L_j\}$ is disjoint from Δ and $t^i_j \cup \{L_j\} \vdash A$, or $\langle t^i_j \cup \{L_j\}$ overlaps Δ and $t^i_j \cup \{R_j\} \vdash A$. The former case is impossible because $A \in \Delta$ and $\langle t^i_j \cup \{L_j\}$. In the latter case, there is $B \in \Delta$ such that $t^i_j \cup \{L_j\} \vdash B$, and $t^i_j \cup \{R_j\} \vdash A$. But $t^i_j \vdash L_j \vee R_j$, so $t^i_j \vdash B \vee A$, which is in Δ since Δ is closed under disjunctions, a contradiction.

Finally, p is definable. We will semi-define multiple relations simultaneously, using multiary neutral fixed-point logic, and by lemma 32, if we can define p in terms of those relations, p is definable.

Let $T(a, i, j)$ hold if $a = \mathsf{enc}(A)$ for some formula A and $A \in t^i_j$. Let $D(a, i, j)$ hold if $a = \mathsf{enc}(A)$ and A is derivable from t^i_j. Let $D_L(a, i, j)$ hold if $a = \mathsf{enc}(A)$ and A is derivable from $t^i_j \cup \{L_j\}$. Let $L(l, i, j)$ hold if l is a list of encodings of formulas drawn from t^i_j, and $L_L(l, i, j)$ hold if l is a list of encodings of formulas drawn from $t^i_j \cup \{L_j\}$. We write down the variable vectors and formulas that will semi-define these relations, in the format: 'Predicate P_i: variables $\vec{x_i}$: formula ϕ_i.' These will be assembled into an expression $E = (\iota(T, D, D_L, L, L_L))(\phi_1, \vec{x_1}, ...\phi_5, \vec{x_5})$. We will reuse variable names for ease of reading, but in E each $\vec{x_i}$ will have different variable names. We can freely use $\in t$ and $\in \Delta$ as predicates because these sets are definable (we really mean, $\in \mathsf{enc}(t)$ and $\in \mathsf{enc}(\Delta)$, of course). We use some other common syntactic abbreviations for readability. Let $L_j \vee R_j$ be the j-th disjunction in our enumeration (which is also definable). Let $\mathsf{Der}(p, s, a)$ mean that p is a list of encodings of formulas which form a derivation sequence, with source encoded by s, deriving a formula encoded by a (Lemma 31).

(1) $T : (a, i, j) : (i, j = 0 \wedge a \in t) \vee (j = 0 \wedge i \neq 0 \wedge \exists k(T(i-1, k, a))) \vee (j \neq 0 \wedge T(a, i, j-1)) \vee (j \neq 0 \wedge a = \mathsf{enc}(L_j) \wedge \forall b(D_L(b, i, j-1) \rightarrow b \notin \Delta))) \vee (j \neq 0 \wedge \exists b(D_L(b, i, j-1) \wedge b \in \Delta) \wedge a = \mathsf{enc}(R_j))$

(2) $D : (a, i, j) : \exists p, s[\mathsf{Der}(p, s, a) \wedge L(s, i, j)]$

(3) $D_L : (a, i, j) : \exists p, s[\mathsf{Der}(p, s, a) \wedge L_L(s, i, j)]$

(4) $L : (l, i, j) : \mathsf{len}(l) = 0 \vee (L(l[0, \mathsf{len}(l) - 1], i, j) \wedge T(l[\mathsf{len}(l)], i, j))$

(5) $L : (l, i, j) : \mathsf{len}(l) = 0 \vee (L(l[0, \mathsf{len}(l) - 1], i, j) \wedge (T(l[\mathsf{len}(l)], i, j) \vee l[\mathsf{len}(l)] = \mathsf{enc}(L_j)))$

By inspection, the intended meanings of T, D, etc. listed above are a solution to E. It remains to prove that they are the unique solution.

We prove this by induction on i, j: Suppose v is a variable assignment and $v(T), v(D), v(D_L), v(L), v(L_L)$ (call these P_i) satisfy the condition that $v[x_i] \in v[P_i]$ iff $\langle \mathbb{N}, S \rangle, v \models \phi_i$.

First of all, for every i, j it's easy to see that $L(l, i, j)$ holds iff l is a list of elements of $\{a \mid T(a, i, j)\}$, and $L_L(a, i, j)$ iff l is a list of elements of $\{a \mid T(a, i, j)\} \cup \{L_j\}$, by induction on $\mathsf{len}(l)$

Then for all i, j, $T(a, i, j)$ holds iff $a \in \mathsf{enc}[t^i_j]$, $D(a, i, j)$ holds iff $t^i_j \vdash A, a = \mathsf{enc}(A)$, etc.

For all i, j: (we will abuse notation and say $\mathsf{enc}[X] \vdash \mathsf{enc}(A)$ when $X \vdash A$):

(a, i, j) is in D iff the defining formula holds, iff there are lists p, s where s is drawn from $\{b \mid T(b, i, j)\}$, such that $\mathsf{Der}(p, s, a)$ holds, iff $\{b \mid T(b, i, j)\} \vdash a$, by lemma 31.

Likewise, $(a, i, j) \in D_L$ iff $\{b \mid T(b, i, j)\} \cup \{L_j\} \vdash a$. So it remains only to prove that for all i, j, $T(a, i, j)$ holds iff $a \in \mathsf{enc}[t^i_j]$, which we now do by induction on i, j.

Base case: $i, j = 0$. Now $(a, i, j) \in T$ iff the defining formula holds, iff its first disjunct holds (the other two assert either i or j is not 0), iff '$a \in t$' holds, iff $a \in \mathsf{enc}[t] = \mathsf{enc}[t^0_0]$.

Inductive cases:

$j > 0$: Then $T(a, i, j)$ holds iff one of the last three disjuncts of the defining formula holds, and these hold just in the same three cases from the definition of t^i_j, because by IH and the claims above, $D(\mathsf{enc}(L_j \vee R_j), i, j-1)$ holds iff t^i_{j-1} derives $L_j \vee R_j$, and so on - note the occurrences of D and D_L all have $j - 1$, so the IH can be applied to them.

$i > 0, j = 0$: then $T(a, i, j)$ holds iff the second disjunct of the defining formula holds, iff for some k, $T(i - 1, k, a)$ holds, iff $a \in \mathsf{enc}[t^{i-1}_k]$ for some k, by IH; this holds iff $a \in \cup_k \mathsf{enc}[t^{i-1}_k] = \mathsf{enc}[\cup_k t^{i-1}_k] = \mathsf{enc}[t^i_0]$, as required.

This concludes the induction.

Therefore, t^i_j is semi-definable, so by lemma 32, it is definable, and thus p is definable (by $\exists i, j[T(a, i, j)]$). □

Lemma 35. *Let S be a definable set of formulas. Then the disjunctive closure of S is definable.*

Proof. We define the relation 'a is a disjunction of depth $\leq n$ in formulas from S', as follows:

$(\iota D)[((n = 0) \wedge S(a)) \vee ((n > 0) \wedge \exists b, c, m, m'[D(m, b) \wedge D(m', c) \wedge m < n \wedge m' < n \wedge (a = f_{\vee}(b, c))]), (n, a)]$

Then the disjunctive closure is defined by $\exists n(D(n, a))$ with free variable a. □

Theorem 36. *The definable formal theories are a sub-canonical set.*

Proof. There are several postulates to prove.

(1) **B** is a definable set. This is lemma 30.

(2) Let A be a fixed formula. Then $\langle A \rangle$ is a definable set because it is just $\{B \mid \{A\} \vdash B\}$, which is defined by $\exists p, s[\mathsf{Der}(p, s, a) \wedge \forall x[\in(x, s) \rightarrow x = a]]$, with free variable a.

(3) Let p be a definable prime theory. Then its dual p^* is definable by $(i_\Phi(a) = 1) \wedge \neg(\in p)(f_\neg(a))$, with free variable a.

(4) Let t, u be definable sets. Then $t \circ u$ is defined by
$\exists i[(\in t)(i) \wedge (\in u)(p_{\rightarrow,1}(i)) \wedge (a = p_{\rightarrow,2}(i))]$, with free variable a.

(5) Let t, u be definable theories and p a definable prime theory. Then there are prime extensions q, r of t, u respectively such that $q \circ u \subseteq p$ and $t \circ r \subseteq p$. To construct q, take the set Δ which is the disjunctive closure of $\{A \rightarrow B \mid A \in t, B \notin P\}$. This set is disjunctively closed. It is also disjoint from s, because if not, there is some $\bigvee(A_i \rightarrow B_i)$ in s such that A_i are in t and B_i are not in p. Then by a theorem of **B**, $\bigwedge A_i \rightarrow \bigvee B_i \in s$. But $\bigwedge A_i \in t$, and so $\bigvee B_i \in p$, so one of the B_i is in p because p is prime. This is a contradiction. Furthermore, Δ is definable. $\{A \rightarrow B \mid A \in t, B \notin P\}$ is defined by:
$\chi_{\rightarrow}(a) \wedge (\in t)(p_{\rightarrow,1}(a)) \wedge \neg(\in p)(p_{\rightarrow,2})$ with free variable a,
and so its disjunctive closure is definable by lemma 35. So all the conditions are in place to use lemma 34, which gives us a definable prime extension q of t disjoint from Δ, which implies that $q \circ u \subseteq p$.

To construct r, we use Δ the disjunctive closure of $\{A \mid \exists B[A \rightarrow B \in t, B \notin p]\}$. This set is disjoint from u because if not, there is some $\bigvee A_i$ in u such that there exist B_i where $(A_i \rightarrow B_i) \in t$ and no B_i is in p. But then $\bigwedge(A_i \rightarrow B_i) \in t$, and it's a theorem of **B** that $\bigwedge(A_i \rightarrow B_i) \rightarrow (\bigvee A_i \rightarrow \bigvee B_i)$, so $(\bigvee A_i \rightarrow \bigvee B_i) \in t$, thus $\bigvee B_i \in p$, so one of the B_i is in p, a contradiction. Δ is disjunctively closed and by a similar argument to the last case it is definable. So lemma 34 constructs the r we want.

(6) For every definable theory t and formula A not in t, there is a definable prime theory p extending t that does not contain A. For, let Δ be the disjunctive closure of $\{A\}$. Then Δ is definable, so we can use lemma 34 if only we can prove Δ is disjoint from t. But for every disjunction $\bigvee A$ of formulas from $\{A\}$, $\bigvee A \rightarrow A$ is a theorem of **B**, so if Δ and t overlapped, t would contain A, a contradiction.

□

Corollary 37. *The definable formal theories, with $P, \ell, \circ, *, , \sqsubseteq, v$ defined as in the canonical model, are a model satisfying exactly the theorems of **B**, and having uncountably many missing theories.*

6. Set-Theoretic Construction

We can also use set theory to obtain a countable subcanonical set. The main advantage of this approach over the last section is that we do not have to re-prove the Lindenbaum lemma in a new setting. The carry-over of known results will be much more automatic. But arguably the model obtained is more interesting than we got from the Lowenheim-Skolem theorem - at least if you think you understand something about countable standard models of ZF.

In this section, all set-theoretic formulas are in first-order logic with equality. This language will be our 'meta-language', and propositional logic our 'object language.' The concept of absoluteness will be very important below. Use the notation $U, v \models \phi$ for 'when the free variables of ϕ are assigned by v, ϕ is really true', where v may assign variables of ϕ to arbitrary sets. To avoid any impression of illegitimacy, we will define this notation outright. We assume our first-order signature has no predicates except $\in$, and only constant and variable terms. Our notion of truth will be relative to an assignment g of constants to some objects.

First, define the interpretation of terms:

(1) $x^{U,v} = v(x)$ when x is a variable
(2) $c^{U,v} = g(c)$ when c is a constant

(1) $U, v \models x \in y$ iff $x^{U,v} \in y^{U,v}$
(2) $U, v \models \phi \wedge \psi$ iff $U, v \models \phi$ and $U, v \models \psi$
(3) $U, v \models \neg\phi$ iff $U, v \not\models \phi$
(4) $U, v \models (\forall x)\phi$ iff for all sets y, $U, v[x\backslash y] \models \phi$

Now say that ϕ is *absolute* for a model S if, for all v mapping into S, we have $S, v \models \phi$ iff $U, v \models \phi$.

Call a model of the $\in$ predicate (and possibly some constants) a *model of set theory*. We will call a model S of set theory *standard* if, for all $x, y \in S$, $x \in^S y$ iff $x \in y$. Call S *transitive* if for all $y \in S$ and all $x \in y$, $x \in S$.

We will sometimes speak of concepts C, by their informal names, being absolute; when we do this, it means 'there is a formula $\phi(x)$ such that $U, [x \mapsto a] \models \phi(x)$ iff C holds of a, and ϕ is absolute' - in other words, some absolute formula formalizes the notion C.

Let the bounded quantifiers $\forall x \in y$ and $\exists x \in y$ be defined as abbreviations for $\forall x[x \in y \to \phi]$ and $\exists x[x \in y \wedge \phi]$. Clearly, $\neg(\forall x \in y)\neg\phi$ is logically equivalent to $(\exists x \in y)\phi$.

Lemma 38. *Let S be a transitive standard model of set theory. Let ϕ be a first-order formula built up from absolute formulas for S using only the bounded quantifiers, $\wedge$ and $\neg$. ϕ is absolute for S.*

Proof. There is some set A of absolute formulas such that ϕ is built up from A using $\neg, \wedge$ and bounded quantifiers. By induction on the formula complexity of ϕ, treating subformulas in A as level 0:

Base Case: $\phi \in A$. Trivial.

Inductive Cases:

$\phi = \psi \wedge \psi'$: $S, v \models \phi$ iff $S, v \models \psi$ and $S, v \models \psi'$, iff $U, v \models \psi$ and $U, v \models \psi'$, iff $U, v \models \psi$.

$\phi = \neg\psi$: $S, v \models \phi$ iff $S, v \not\models \psi$ iff $U, v \not\models \psi$ iff $U, v \models \neg\psi$.

$\phi = (\forall x \in y)\psi$: $S, v \models \phi$ iff for all $z \in^S v(y)$, $S, v[x\backslash z] \models \psi$, iff for all $z \in v(y)$, $S, v[x\backslash z] \models \psi$ (because S is standard transitive), iff for all $z \in v(y)$, $U, v[x\backslash z] \models \psi$, iff $U, v \models (\forall x \in y)\psi$. □

Note that in a standard model, $x \in y$ and $x = y$ are absolute. $x \in c$, $c \in x$, $c \in d$ are also absolute if S happens to assign c, d to $g(c), g(d)$.

Corollary 39. *Let S be a standard transitive model. If the only logical constructors in ϕ are $\wedge, \vee, \neg$ and bounded quantifiers, and for all constants c appearing in ϕ, $g(c) = c^S$, then ϕ is absolute for S.*

Define the deep bounded quantifiers $(\forall x \in^n y)$ for each n as abbreviations for

- If $n = 1$: $(\forall x \in^1 y)\phi = (\forall x \in y)\phi$
- If $n > 1$: $(\forall x \in^n y)\phi = (\forall m \in^{n-1} y)(\forall x \in m)\phi$

In other words, 'for all x nested n-deep inside y'. Define $(\exists x \in^n y)$ analogously. Note that these abbreviations only use bounded quantifiers, so lemma 38 applies to formulas using them.

For a set U, let a *rule over U* be a pair (B, H) where $B, H \subseteq U$. Say a subset U' of U is *closed under B, H* if $B \nsubseteq U'$, or $H \subseteq U'$. U' is closed under a set of rules if it is closed under each of the rules in the set.

Most of the following tedious list is standard background, but note the last few cases:

Lemma 40. *For any transitive standard model of set theory, the following are absolute:*

(1) $x = \{y_1, ...y_n\}$
(2) *for some y, z, $x = (y, z)$*
(3) $x = (y, z)$
(4) *R is a relation*
(5) *f is a function*
(6) *f is an injective function*
(7) *f has range (image) R*
(8) *f has domain D*
(9) $f(x) = y$
(10) $x \subseteq y$
(11) *x and y are disjoint*
(12) $x = \emptyset$
(13) $x = S(y)$
(14) $x = y \times z$
(15) $x = \omega$
(16) *R is a set of rules over U*
(17) *U' is closed under the set of rules R over U*
(18) *R is a set of rules over U, $U' \subseteq U$ and X is the smallest superset of U' closed under R*

Proof. Most of these we prove by just displaying formulas that define them, which you can inspect to see that they only use bounded quantifiers and absolute predicates from higher up the list. The result follows for these cases by lemma 38.

(1) $(\forall z \in x)[z = y_1 \vee ... \vee z = y_n] \wedge (y_1 \in x) \wedge ... \wedge (y_n \in x)$

(2) This is to say $x = \{\{y\}, \{y, z\}\}$ using the standard encoding of pairs, so we can write $(\exists a_y \in x)(\exists a_{yz} \in x)(\exists y \in a_y)(\exists z \in a_{yz})[a_y = \{y\} \wedge a_{yz} = \{y, z\}]$

(3) Same as above, omitting the quantifiers on y, z.

(4) $(\forall x \in R)[\exists y, z[x = (y, z)]]$

(5) f is a relation and $(\forall p, q \in f)(\forall y, z \in^2 p)(\forall y', z' \in^2 q)[(p = (y, z) \wedge q = (y', z') \wedge (y = y')) \rightarrow z = z']$

(6) f is a function and $(\forall p, q \in f)(\forall y, z \in^2 p)(\forall y', z' \in^2 q)[(p = (y, z) \wedge q = (y', z') \wedge (z = z')) \rightarrow y = y']$

(7) f is a relation and $(\forall p \in f)(\forall y, z \in^2 p)[p = (y, z) \rightarrow z \in R] \wedge (\forall z \in R)(\exists p \in f)(\exists y \in^2 p)[p = (y, z)]$

(8) f is a relation and $(\forall p \in f)(\forall y, z \in^2 p)[p = (y, z) \rightarrow y \in D] \wedge (\forall y \in D)(\exists p \in f)(\exists z \in^2 p)[p = (y, z)]$

(9) $(\exists f' \in f)(\exists a, b \in^2 f')[f' = (a, b) \wedge a = x \wedge b = y]$

(10) $(\forall z \in x)[z \in y]$

(11) $(\forall z \in x)[\neg(z \in y)]$

(12) $(\forall z \in x)[\neg(z \in x)]$

(13) $(\forall z \in x)[(z \in y) \vee z = \{y\})] \wedge (\forall z \in y)[z \in x] \wedge (\exists z \in x)[z = \{y\}]$

(14) $(\forall e \in x)(\exists e_y \in y)(\exists e_z \in z)[e = (e_y, e_z)] \wedge (\forall e_y \in y)(\forall e_z \in z)(\exists e \in x)[e = (e_y, e_z)]$

(15) $(\exists e \in x)[e = \emptyset] \wedge (\forall z \in x)(\exists z' \in x)[z' = S(z)] \wedge (\forall z \in x)[(z = \emptyset) \vee (\exists z' \in x)[z = S(z')]]$. To see that this does define ω in the real universe, note that if x satisfies this formula it contains $0 = \emptyset$ and is closed under successors, so x contains all natural numbers. Suppose for contradiction x contains some z_0 that is not a natural number. Then z_0 is not $\emptyset$, so by the third clause, there is $z_1 \in x$ where $z_0 = S(z_1)$. But if z_1 is a natural number, z_0 is a natural number, so z_1 is not a natural number. And by the nature of successors, $z_1 \in z_0$. Repeating this process constructs an infinite descending chain of sets, contradicting the Foundation axiom.

(16) $(\forall r \in R)(\exists H, B \in^2 r)[r = (H, B) \wedge H \subseteq U \wedge B \subseteq U]$

(17) R is a set of rules over U, and $(\forall r \in R)(\forall H, B \in^2 R)[(r = (H, B)) \wedge (H \subseteq U') \rightarrow (B \subseteq U')]$

(18) For this one, our strategy is different. We prove that, if S is a transitive standard model, for all U', R in S, the smallest superset of y closed under R is indeed in S. Then, since being a closed extension is absolute, the smallest closed extension of U' *in* S and the smallest closed extension of U' *among all sets* are the same set - so although it contains an unbounded quantifier, the formula 'X is closed under R and $U' \subseteq X$ and $(\forall Z)[(U' \subseteq Z) \wedge Z$ is closed under $R] \rightarrow X \subseteq Z]$ is absolute.

Proof of claim: For $R, U \in S$, consider the 'step' operator $T : A \mapsto \{B \mid \exists H(H, B) \in R, X \subseteq A\}$, defined on subsets of U in S. T is also in S, because if A is in S, so is $T(A)$, as it is the subset of U defined by $\{B \mid \exists H[(H, B) \in R \wedge H \subseteq A]\}$, so $T(A) \in S$ by the axiom of restricted comprehension. Thus, all the pairs $(A, T(A))$ are in the set $\mathcal{P}^S(U) \times \mathcal{P}^S(U)$, which is in S. Then by comprehension again, (the graph of) T is a member of S. Now define the function $A_{(\cdot)} : \omega^S \rightarrow \mathcal{P}^S(U)$ given by $A_0 = A, A_{n+1} = T(A_n)$, which is in S by the recursion theorem. We can then define $\{x \mid \exists(n \in \omega^S)[x \in A_n]\}$ as a subset of U by comprehension.

But since ω is absolute, $\omega^S = \mathbb{N}$, so this is exactly $\bigcup_{n\in\mathbb{N}} T^n(A)$, the set of objects contained in every superset of A closed under R.

□

We will also use a variant of the "Σ_1" lemma.

Lemma 41. *Let ϕ be absolute in a model S of set theory. Then if $S, v \models \exists\vec{x}\phi$, $U, v \models \exists\vec{x}\phi$.*

Proof. If $S, v \models \exists\vec{x}\phi$, then $S, v[\vec{x}\backslash\vec{z}] \models \phi$ for some $\vec{z}$, but then by absoluteness $U, v[\vec{x}\backslash\vec{z}] \models \phi$, so $U, v \models \exists\vec{x}\phi$. □

6.1. **Set Theory with Logical Vocabulary.** Let ZF+L (ZF with a distinguished logical vocabulary) be the first order theory which has as its language a binary predicate $\in$ and constant symbols Σ, Σ^*, $f_\wedge, f_\vee, f_\rightarrow, f_\neg$, and countably infinitely many constant symbols A_n. It is axiomatized by the axioms of ZF, plus the following:

(1) $\Sigma \subseteq \Sigma^*$, Σ is infinite,
(2) $\wedge, \vee, \rightarrow: (\Sigma^*)^2 \rightarrow \Sigma^*$,
(3) $\neg : \Sigma^* \rightarrow \Sigma^*$,
(4) $f_\wedge, f_\vee, f_\rightarrow, f_\neg$ are injective and have disjoint ranges,
(5) the ranges of $f_\wedge, f_\vee, f_\rightarrow, f_\neg$ are disjoint from Σ,
(6) Σ^* is the smallest set containing Σ and closed under $f_\wedge, f_\vee, f_\rightarrow, f_\neg$
(7) $A_n \in \Sigma$ for all n
(8) $A_i \neq A_j$ for all $i \neq j$

Let a model M of this theory be called *standard* if its domain consists of sets and $\in^M$ is the actual membership relation on those sets. Call M *transitive* if, whenever $y \in M$ and $x \in y$, $x \in M$. Call a model *well-labeled* if $\Sigma^M = \{A_n^M \mid n \in \mathbb{N}\}$.

Lemma 42. *If there exists a countable transitive standard model of ZF, there exists a countable transitive standard well-labeled model of ZF+L.*

Proof. It is well-known that we can prove (without using the axiom of choice) that some sets and functions do exist satisfying the axioms 1-6 above, e.g. by making formulas finite sequences of symbols from the alphabet, connective symbols, and parentheses, and coding all these symbols with natural numbers. This existence is thus a theorem of ZF; call it $\exists\vec{x}\phi$ where $\vec{x}$ has six variables. So, let S be a countable transitive standard model of ZF. $S \models \exists\vec{x}\phi$, so there is some assignment v of $\vec{x}$ into S satisfying ϕ, that is, axioms 1-6. Now extend S to a model S' by letting $\Sigma^{S'}, (\Sigma^*)^{S'}, f_\wedge^{S'}$, etc. be the elements of $v[\vec{x}]$. $v(\Sigma)$ is actually infinite, and countable because S is countable. For this larger signature, we will define absoluteness using $g(\Sigma) = \Sigma^{S'}, g(\Sigma^*) = (\Sigma^*)^{S'}$, etc.

Axioms 3-6 are true of $v(\Sigma), v(\Sigma^*)$, etc: For axioms 2 and 3 this is obvious from lemma 40, because these are absolute in S. For axioms 4 and 5, we use lemma 41. $\phi(R, f, \Sigma) =$'R is the range of f and R is disjoint from Σ' is absolute, using lemma 40, and therefore since $S, v \models \exists R\phi(R, f, \Sigma)$, then also $U, v \models \exists R\phi(R, f, \Sigma)$, that is, there is a set R that is really the range of $v(f)$ and it is really disjoint from $v(\Sigma)$. Axiom 4 is similar. For axiom 6, define the set of rules $R = \{(\{x, y\}, \{f(x, y)\}\} \mid x, y \in \Sigma^*, f \in \{v(f_\wedge), v(f_\vee), v(f_\rightarrow), v(f_\neg)\}\}$. This is clearly definable by a formula, and its interpretation in S' is a set of rules over Σ^*. The claim that Σ^* is the smallest subset of Σ^* containing Σ and closed under these rules is absolute, so $(\Sigma^*)^{S'}$ really is the smallest such set. Furthermore, the formula $(\forall r \in R)(\exists H, B \in^2 R \exists x, y, z \in \Sigma^*)[r = (H, B) \wedge H = \{x, y\} \wedge B = \{z\} \wedge f_\wedge(x, y) = z] \wedge \forall x, y, z \in \Sigma^*[(f(x, y) = z) \to (\exists r \in R)(\exists H, B \in^2 R)[(r = (H, B) \wedge H = \{x, y\} \wedge B = \{z\}]]$ is absolute and expresses 'R contains exactly rules of the form $\{x, y\}, f(x, y)$' - this can be easily extended to cover all four functions $f_\wedge, f_\vee, f_\to, f_\neg$. Therefore closure under these rules is really equivalent to closure under the f's, so $(\Sigma^*)^{S'}$ is really the closure of Σ^S under the $f^{S'}$-s.

Now S' satisfies 1-6, which are also actually true of $\Sigma^{S'}$, ...

Then extend S' to a model S'' evaluating the symbols A_n: Choose any bijection $i : \mathbb{N} \to \Sigma^S$, and let $A_n^{S''} = i(n)$. So S'' satisfies 7-8 and is well-labeled. □

From now on, we fix a particular standard transitive countable well-labeled model S of ZF+L, where the constants A_n in the signature are the propositional vocabulary for our propositional formulas. We fix an encoding of propositional formulas A as elements of S:

(1) If A is an atom, $\mathsf{enc}(A) = A^S$
(2) If $A = B \wedge C$, $\mathsf{enc}(A) = f_\wedge^S(\mathsf{enc}(B), \mathsf{enc}(C))$
(3) If $A = B \vee C$, $\mathsf{enc}(A) = f_\vee^S(\mathsf{enc}(B), \mathsf{enc}(C))$
(4) If $A = B \to C$, $\mathsf{enc}(A) = f_\to^S(\mathsf{enc}(B), \mathsf{enc}(C))$
(5) If $A = \neg B$, $\mathsf{enc}(A) = f_\neg(\mathsf{enc}(B))$

Lemma 43. *enc is injective and onto $(\Sigma^*)^S$.*

Proof. Injective: Let A, B be different formulas. We will do induction on the sum of formula complexities of A, B. Base case: If A, B are both atoms, $\mathsf{enc}(A) \neq \mathsf{enc}(B)$ because of axiom 8.

Inductive case: Many cases are trivial. If only one of A, B is an atom, their encodings are distinct by axiom 5. So let both A, B be non-atoms. Then if they have different main connectives, their encodings are distinct by axiom 4. So we only need to use the IH when the main connectives of A, B are the same. Suppose (wlog) that both are $\wedge$, so $A = A' \wedge A''$, $B = B' \wedge B''$. By axiom 4, $f_\wedge^S$ is injective, so if $\mathsf{enc}(A) = \mathsf{enc}(B)$, we have $(\mathsf{enc}(A'), \mathsf{enc}(A'')) = (\mathsf{enc}(B'), \mathsf{enc}(B''))$. But either $A' \neq A''$ or $B' \neq B''$, or else A, B would be the same formula, so we have contradicted the IH.

Onto: For contradiction, let $x \in (\Sigma^*)^S$ but not an encoding of a formula. Then since S is well-labeled, $x \notin \Sigma^S$. Now the set $(\Sigma^*)^S \setminus \{x\}$ contains Σ^S and is closed under $f_\wedge^S$, ..., so $(\Sigma^*)^S$ was not the smallest such set, a contradiction. □

From now on we will not distinguish between a formula and its encoding when no confusion can arise. Let the encoding of a set of formulas (such as a theory) be its enc-image. The countable **B** submodel we extract will consist of the theories whose encodings happen to be members of S.

Now we can say things like this:

Lemma 44. *For any object-language formulas $A_1, ...A_n$, the statement 'for some substitution σ, $B_1, ...B_n = \mathsf{enc}(A_1\sigma), ...\mathsf{enc}(A_n\sigma)$' (where B_i are regarded as free variables) is absolute.*

Proof. By induction on the total formula complexity of $A_1, ...A_n$, we show that there is a formula $\phi_{A_1,...A_n}(B_1, ...B_n)$ over the language of ZF+L defining this concept and it is absolute.

Base Case: complexity = 0. Then all A_i are atoms. $\phi_{A_1,...A_n} = (\bigwedge_i B_i \in \Sigma^*) \wedge \bigwedge_{(i,j)\in D} (B_i = B_j)$, where D is the finite set of (i, j) such that $A_i = A_j$. Any B_i's satisfying this are indeed encodings of formulas (since enc is onto Σ^*), and for $A_i = A_j$, B_i, B_j encode the same formula - thus $\sigma : A_i \mapsto \mathsf{enc}^{-1}(B_i)$ is the desired substitution. Conversely, if $B_i = \mathsf{enc}(A_i\sigma)$ for some σ, the formula $\phi_{A_1,...A_n}$ clearly holds. This formula is absolute, as it is quantifier-free.

Inductive Case: One of the A_i has a main connective. Suppose without loss of generality it is $\wedge$, so $A_i = A_{i1} \wedge A_{i2}$. Then by IH there is an absolute formula $F(B_1, ...B_{i-1}, B_{i1}, B_{i2}, B_{i+1}, ...B_n)$ defining encodings of substitution instances of $A_1, ...A_{i-1}, A_{i1}, A_{i2}, A_{i+1}, ...A_n$.

Let $\phi_{A_1,...A_n}(B_1, ...B_n)$ be $\exists B_{i1}, B_{i2} \in \Sigma^*[F(B_1, ...B_{i-1}, B_{i1}, B_{i2}, ...B_n) \wedge B_i = f_\wedge(B_{i1}, B_{i2})]$.

Clearly this defines the desired concept, and it is still absolute because the quantifier is bounded by Σ^*. □

Remember that we have identified formulas with their encodings, so it makes sense to say 'the logic generated by a Hilbert system H' is a subset of $(\Sigma^*)^S$.

Lemma 45. *For any finite Hilbert system H, 'the logic generated by H' is absolute, and defines a set that exists in S.*

Proof. There are finitely many axioms $\{A_1, ...A_n\}$ and rules $\{R_1, ...R_m\}$ of which all axioms and rules of H are substitution instances. Then we can define the set of axioms of H by

$\bigvee_1^n [x \text{ is a substitution instance of } A_i]$,

which is expressible and absolute by lemma 44.

Each Hilbert rule $R_i = \dfrac{B_{i,1}, ...B_{i,k}}{H_i}$ can be thought of as the 'rule' (as defined above) $(\{B_{i,1}, ...B_{i,k}\}, \{H_i\})$. Now the substitution instances R' of this rule can be defined by the formula

$(\exists B, H \in^2 R')(\exists B'_{i,1}, ...B'_{i,k} \in B)(\exists H'_i \in H)[R' = (H, B) \wedge H = \{H'_i\} \wedge B = \{B'_{i,1}, ...B'_{i,k}\} \wedge B'_{i,j}, H'_i$ are encodings of a substitution instance of $B_{i,j}, H_i]$.

Thus we can prove the existence of the set of rules of H (up to encoding) using the comprehension axiom. Now the logic of H is the smallest set containing the axioms and closed under the rules, in the same sense that we dealt with above - so by the final point of lemma 40, the logic is definable and absolute. □

Lemma 46. *Being a formal theory is absolute.*

Proof. A set of formulas is a theory iff it is closed under the following set of rules: $\{(\{A, B\}, \{f_\wedge(A, B)\}) \mid A, B \in \Sigma^*\} \cup \{(\{A\}, \{B\}) \mid f_\to(A, B) \in \mathbf{B}\}$, where $\mathbf{B}$ is the logic $\mathbf{B}$, which is definable by an absolute formula by the last lemma. Now each of these rules is in S, as can be proven with the pairing axiom, because they are finite. By comprehension, the set of these rules is in S. Furthermore, this set is absolute (because quantifiers may be bounded to Σ^*). So, the predicate '$T \subseteq \Sigma^*$ and is closed under these rules' is absolute, by lemma 40. □

Lemma 47. *Being a prime set is absolute.*

Proof. 'X is prime' is defined by $(\forall A \in X)(\forall B, C \in \Sigma^*)[A = f_\vee(B, C) \to (B \in X \vee C \in X)]$, which has only absolute predicates and bounded quantifiers. □

Theorem 48. *The set of formal theories that are in S are a subcanonical set.*

Proof. Let C be the interpretation of the concept 'formal theory' in S. By lemma 46, this is the set of all formal theories in S. We prove each of the conditions for subcanonicalness. Our strategy is this: each condition is of the form $\forall\vec{x}[R'(\vec{x}) \to \exists t R(t, \vec{x})]$, where R, R' are absolute properties. Each condition is also a theorem of ZF (since the actual set of all theories really satisfies all the conditions to be subcanonical, and this is provable in ZF).

Suppose we have objects $\vec{x}$ in S really satisfying R'. That is, letting $v[\vec{y}] = \vec{x}$, $U, v \models R'(\vec{y})$. Then by absoluteness, $S, v \models R'(y)$. Since S is a model of ZF, $S, v \models \exists t R(t, \vec{y})$. Thus for some extension v' of v to t, $S, v' \models R(t, \vec{y})$. Therefore $U, v' \models R(t, \vec{y})$, which means $R(v'(t), \vec{x})$ really holds, and $v'(t) \in S$. Thus the condition $\forall\vec{x}[R'(\vec{x}) \to \exists t R(t, \vec{x})]$ really holds over the sets that lie in S.

For each case we will merely prove that the conditions R, R' are absolute, usually by showing a formula defining them, which can be seen to be absolute by the above lemmas.

(1) **B** has a finite Hilbert system, and therefore is defined by an absolute formula.
(2) For each A, $\langle A \rangle$ is in S. For, as in lemma 46, we can define an absolute set of rules, closure under which defines theoryhood, and we can define A, and by the last point of lemma 40, the smallest theory containing A is absolute.
(3) The relation of $t \circ s$ to t, s is defined by the formula $(\forall B \in \Sigma^*)(B \in x \leftrightarrow (\exists A \in \Sigma^*)[A \in s \wedge (f_{\to}(A, B) \in t))]$, which is absolute by its form.
(4) Similarly, the relation of p^* to p is defined by $(\forall A \in \Sigma^*)(A \in x \leftrightarrow f_{\neg}(A) \notin p)$.
(5) The relation of t, u, p is definable by: p is prime, and $(\forall I \in t)(\forall A \in u)(\forall B \in \Sigma^*)[I = f_{\to}(A, B) \wedge A \in u \to B \in p]$. And the relation of t, u, q, p is definable by: q is prime, $t \subseteq q$, and $(\forall I \in q)(\forall A \in u)(\forall B \in \Sigma^*)[I = f_{\to}(A, B) \wedge A \in u \to B \in p]$. Likewise for t, u, r, p. These are all absolute formulas by inspection.
(6) The relation of t to A is simply that t is a theory and $A \notin t$. The relation of t, A, p is that p is a prime theory and $t \subseteq p$ and $A \notin p$, all of which are absolute.

□

Conclusion

We have proven that there are frame models of **B** that differ from the canonical model in two important ways, indicating that the semantics of **B** is justified not only by reference to the space of **B** formal theories, but to a somewhat more general family of theory spaces. In fact we can remove, not just some theories, but most of the theories, from the canonical model, and not invalidate any theorems of **B**. We can also have points in theory spaces that do not look like formal theories, because they are not individuated by the set of formulas they satisfy. Are such twin points a problem for the interpretation of points as 'theories'? Should a realistic frame semantics make them impossible? We have no strong opinion on this, and leave it for future discussion.

Most of the methods in this paper can apply to models for stronger relevant logics as well (for the argument of section 4, this is as easy as adding to BMod some first-order axioms translating additional semantic postulates).

Unfortunately, the most explicit 'small' model of **B** we obtained still contains all **B** theories that are in the recursive hierarchy, which are probably all the theories most people will ever need. In that sense this model still looks pretty canonical. Open problems: is there a model M of **B** such that the theories $\{\Gamma \mid \exists t \in T_M[\Gamma = \bar{t}]\}$ lie in a finite level of the recursive hierarchy? On the other hand, are there current philosophical or mathematical theories that, idealized as propositional theories, are not neutral fixed-point definable? If there are any, we have shown that the existence of theories of this kind has no impact on the (proposition-level) rules of theory-building (theorems of **B**).

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References

[1] R. Routley, V. Plumwood, R. K. Meyer, and R. T. Brady, *Relevant logics and their rivals*. Ridgeview Publishing Company, 1982.

[2] S. A. Kripke, "A completeness theorem in modal logic," *Journal of Symbolic Logic*, vol. 24, pp. 1 – 14, 1959. [Online]. Available: https://api.semanticscholar.org/CorpusID:44300271

[3] J. Barwise, "Constraints, channels and the flow of information," in *Situation Theory and its Applications Vol. 3*, P. Aczel, D. Israel, Y. Katagiri, and S. Peters, Eds. CSLI Publications, 1993, pp. 3–27.

[4] E. Mares, *Relevant Logic: A Philosophical Interpretation*. Cambridge University Press, 08 2007.

[5] J. Beall, R. Brady, J. Dunn, A. Hazen, E. Mares, R. Meyer, G. Restall, D. Ripley, J. Slaney, and R. Sylvan, "On the ternary relation and conditionality," *Journal of Philosophical Logic*, vol. 41, pp. 1–18, 06 2012. [Online]. Available: https://www.researchgate.net/publication/225360998_On_the_Ternary_Relation_and_Conditionality

[6] K. Fine, "Models for entailment," *Journal of Philosophical Logic*, vol. 3, no. 4, pp. 347–372, 1974. [Online]. Available: http://www.jstor.org/stable/30226092

[7] S. Logan, "Nondeterministic and nonconcurrent computational semantics for BB+ and related logics," *Journal of Logic and Computation*, pp. 1–20, forthcoming. [Online]. Available: https://philpapers.org/rec/LOGNAN

[8] A. Tedder, "Situations, propositions, and information states (1st edition)," in *Relevance Logics and other Tools for Reasoning: Essays in Honor of J. Michael Dunn*, K. Bimbo, Ed. College Publications, 2023, pp. 410–426. [Online]. Available: https://philarchive.org/rec/TEDSPA

[9] G. Restall and S. Standefer, "Collection frames for distributive substructural logics," *The Review of Symbolic Logic*, vol. 16, no. 4, p. 1120–1157, 2023.

[10] R. T. Brady, "Some concerns regarding ternary-relation semantics and truth-theoretic semantics in general," *IfCoLog Journal of Logics and Their Applications*, vol. 4, no. 3, pp. 755–781, 2017. [Online]. Available: https://philarchive.org/rec/BRASCR-2

[11] S. A. Logan, "The universal theory building toolkit is substructural," in *Alasdair Urquhart on Nonclassical and Algebraic Logic and Complexity of Proofs*, I. Duntsch and E. D. Mares, Eds. Springer Verlag, 2022, pp. 261–285. [Online]. Available: https://philpapers.org/rec/ALLTUT

[12] J. Slaney, "A general logic," *Australasian Journal of Philosophy*, vol. 68, no. 1, pp. 74–88, 1990. [Online]. Available: https://philpapers.org/rec/SLAAGL

[13] A. Dawar and Y. Gurevich, "Fixed point logics," *The Bulletin of Symbolic Logic*, vol. 8, no. 1, pp. 65–88, 2002. [Online]. Available: http://www.jstor.org/stable/2687735