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NEGATED IMPLICATIONS IN CONNEXIVE RELEVANT LOGICS

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ABSTRACT. Connexive expansions of relevant logics tend to prove every negated implication formula. In this paper I discuss why they tend to satisfy this unsavoury property, and discuss avenues by which it can be avoided, providing logics which stand as proofs of concept that these avenues can be made to work.

1. INTRODUCTION

Relevant logics and connexive logics are like (occasionally squabbling) siblings. Originally introduced for similar motivations (to critique the classical theory of material implication, replacing it with one taking account of the need for some *connection* between antecedents and consequents), they were often studied in similar places and by similar people.[1] Furthermore connexivists and relevantists agreed on a number of commitments, such as something like paraconsistency. They have, however, put their focus on different properties of logics as expressing the desired connection.

Relevantists, as their name suggests, push *relevance* as the key requirement, so that in order for A to imply (or entail) B (i.e. for $A \to B$ to be (logically) true), it must be that these are relevant to each other (as expressed in Belnap's VARIABLE SHARING principle, to be stated below). Connexivists focus their attention not on implication $\to$ itself, but on its interaction with negation $\neg$, in particular requiring the following theses to be valid:

ARISTOTLE	$\neg(A \to \neg A)$	$\neg(\neg A \to A)$
BOETHIUS	$(A \to B) \to \neg(A \to \neg B)$	$(A \to \neg B) \to \neg(A \to B)$

The constraints amount to requiring that no formula implies, or is implied by, its own negation (ARISTOTLE) and furthermore that no formula implies

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[1]In the 1960s, focus was put on connexive logics by Richard Angell [3] and Storrs McCall [23], both of whom were in contact with advocates of relevant logics including Anderson and Belnap [2] and their students.

any contradictory pair of formulas (BOETHIUS). The upshot of this is, in a sense, to keep formulas and their negations *separate* as far as implication goes.[2] As it turns out, one easy way to do something like the above is to treat implication as a sort of *biconditional*, leading McCall [23] to place as a further constraint that the following be *invalid*:

SYMMETRY $\qquad (A \to B) \to (B \to A)$

Connexive logics, then, can be understood to be those in a language including at least $\to$ and $\neg$ as connectives which validate ARISTOTLE and BOETHIUS while invalidating SYMMETRY.

Relevant logics, on the other hand, do not tend to be presented as logics validating some theses, but rather as those invalidating some theses, such as those which contravene VARIABLE SHARING. There is, however, more or less widespread agreement on principles which those logics called relevant tend to share. One locus of such agreement comes out of the frame semantics introduced by Sylvan (né Routley) and Meyer [31, 33], and generalised outward to *gaggle theory* in work by Dunn [10] and his collaborators, most notably Bimbó [6]. The distinctive feature of this semantic framework is in requiring operations to have *tonicity types*, specifying whether they preserve or anti-preserve valid implications in some or all of their argument places. This kind of specification of connective meanings places entailment (or provable implication) at the heart of the theory of the meaning of the logical constants (as against, say, truth/falsity, which shows up downstream), which is a move that fits the relevant logic enterprise well. One place where this matters to recent debates, and for some of the discussion in this paper, is in taking satisfaction of something like *contraposition* to be a key property of negation. This has proved to be a point of contention between relevantists and connexivists, broadly understood.[3]

Despite these differences, there have been a handful of attempts to combine these research projects, and to study *connexive relevant logics*. Some early results in this direction are in the work of Mortensen [25] and related work in [33]. There it was noted, as I'll discuss below, that we can

[2] Variants on these relations can be found in the connexive literature. One notable variant is Kapsner's [18] *strong connexivity* requirement, amounting to the claims that implications of the form $A \to \neg A$ and $\neg A \to A$ be unsatisfiable, and furthermore that the pair $\langle A \to B, A \to \neg B \rangle$ not be jointly satisfiable. The algebraic semantics of [37] gives a precise sense in which these requirements involve keeping the values of formulas and their negations separate.

[3] A recent debate on the key properties of negation which touches on this theme is that between Berto and Restall [5] and De and Omori [9]. I'll not wade into the details of this debate, except to briefly tip my hand in favour of the former – this, as we'll see, motivates some of my focus in this paper.

extend at least some relevant logics with connexive theses while obtaining reasonable, and interesting, results. There is an odd property which results from adding connexive principles to relevant logics in almost all cases – doing so results in every negated implication (i.e., every formula of the form $\neg(A \to B)$) being valid. While this unsavoury result has raised eyebrows, it interestingly turns out not to cause too much damage in some of the logics in question (as we'll see). It is, however, pretty unsavoury, and so worth avoiding. The aim of this paper is to lay out the options for how we might add connexive theses to relevant logics while avoiding this consequence, and providing proofs of concept that it can be done in accordance with the options I'll consider. In doing so, I'll spend most of my time interested in logics with some form of contraposition, in accordance with the kinds of commitments accepted in the (broadly) relevantist camp.

The rough plan of the paper is as follows. In §1 I'll lay out some definitions and other preliminaries, including defining the class of relevant logics (and fragments thereof) in which I'll be interested. In §2, I'll recapitulate, from [33], the proof that almost all connexive relevant logics have $\neg(A \to B)$, and contexts in which these logics are, and are not, trivial. In §3, I'll lay out the options for avoiding this unsavouriness (and triviality), and go through the options: *ill-contraposing* logics, *ill-affixing* logics, and *intensional* logics. I conclude in §4 with some remarks on future directions.

2. Preliminaries

2.1. **Some Definitions.**

Definition 2.1 (Languages)**.** A language is a the absolutely free algebra $\langle Fm, \{\otimes_i\}_{i\in I}\rangle$ constructed out of a countably infinite set At_{Fm} of *atomic formulas* by application of the connectives (operation symbols) $\{\otimes_i\}_{i\in I}$, each of some finite arity. The set of the arities of $\{\otimes_i\}_{i\in I}$ is the *similarity type* of the language.[4] I'll use lower case letters from the middle of the Latin alphabet (such as $p, q, r, \dots$) as metavariables over At_{Fm}, and upper case letters from the beginning of that alphabet (such as $A, B, C, \dots$) as metavariables over Fm.

All the languages in question here contain at least the unary operation $\neg$ and the binary operation $\to$, but may also include some of the following binary operations:

- $\wedge$ (*lattice* or *extensional* conjunction)
- $\vee$ (*lattice* or *extensional* disjunction)
- $\circ$ (*intensional* conjunction, or *fusion*)
- $+$ (*intensional* disjunction, or *fission*)

[4] A similar remark in the definition of *similarity type* applies for any algebra.

The set of *endomorphisms* on Fm, written $EndFm$, is the collection of functions $\sigma : Fm \longrightarrow Fm$ which are *homomorphic*, i.e. those such that for each n-ary operation $\otimes$ and collection $\{A_i\}_{i\leq n} \subseteq Fm$:

$$\sigma \otimes (A_1, \ldots, A_n) = \otimes(\sigma A_1, \ldots, \sigma A_n)$$

Alternately, these are called *substitutions*.

A *homomorphism* from Fm to an algebra of the same similarity type $\mathcal{M}$ is a function satisfying the displayed condition.

Definition 2.2 (Logics)**.** $\mathbf{L} \subseteq Fm$ is a *logic* just in case:

(1) For any $\sigma \in EndFm$, $\sigma\mathbf{L} = \{\sigma A \mid A \in \mathbf{L}\} \subseteq \mathbf{L}$.
(2) For any $A, B \in Fm$, if $\{A \to B, A\} \subseteq \mathbf{L}$ then $B \in \mathbf{L}$.

Definition 2.3 (Axiomatic Extensions of Logics)**.** Given a logic $\mathbf{L}$ and a collection $X \subseteq Fm$ (of "axioms"), call $\mathbf{L} \oplus X$, the *axiomatic extension* of $\mathbf{L}$ by X, the least logic extending $\mathbf{L} \cup X$.[5]

Definition 2.4 (Fragments of Logics)**.** Given a logic $\mathbf{L}$ in a language Fm such that $Fm' \subset Fm$ is another language: the *fragment* of $\mathbf{L}$ relative to Fm' (or the Fm'*-fragment* of $\mathbf{L}$) is $\mathbf{L} \cap Fm'$.[6] Where Fm' has operations $\{\otimes_i\}_{i\in I}$, I'll write this as $\mathbf{L}_{\{\otimes_i\}_{i\in I}}$.

By the *positive fragment* of a logic, I mean that without the connective $\neg$, and write this $\mathbf{L}^+$.

For any system $\mathbf{L}$, I'll call $\mathbf{L}_{\to,\neg}$ the *intensional fragment*.

Definition 2.5 (Contradictory Logics)**.** A logic $\mathbf{L}$ is *contradictory* just in case for some $A \in Fm$, $\{A, \neg, A\} \subseteq \mathbf{L}$.

Definition 2.6 (Trivial Logics)**.** A logic $\mathbf{L}$ is *trivial* just in case $\mathbf{L} = Fm$.

The following definitions are stated in generality, though I'll be dealing with logics in a fairly limited language here, as the possibilities for extensions of the language are significant here.

Definition 2.7 (Matrices)**.** A *matrix* is a tuple $\mathcal{M} = \langle \mathcal{M}, \{\otimes_i\}_{i\in I}, D\rangle$ where $\langle \mathcal{M}, \{\otimes_i\}_{i\in I}\rangle$ is an algebra and $\varnothing \neq D \subseteq \mathcal{M}$ is a set of *designated* elements.

[5]The definition of "Logic" just requires these to be closed under the rule form of modus ponens:

$$(\text{rMP})\ \frac{A \to B \qquad A}{B}$$

When I'm considering extensions of logics which are axiomatised here with other rules, I'll suppose that logical extensions thereof are also closed under those extra rules – this is significant in this very section, where I give definitions for systems like **R**.

[6]That this is a logic is easy to verify, noting that we require all languages here to include $\to$ among its operations.

Definition 2.8 (Matrices as Models)**.** Given a logic $\mathbf{L}$ in the language Fm, $\mathcal{M}$ is a *model* of $\mathbf{L}$ just in case $\langle \mathcal{M}, \{\otimes_i\}_{i \in I}\rangle$ has the same similarity type as Fm, and every homomorphism $h : Fm \longrightarrow \mathcal{M}$ is such that if $A \in \mathbf{L}$ then $hA \in D$. In such a circumstance, $\langle \mathcal{M}, h\rangle$ is called the model of $\mathbf{L}$.

I introduce these last two definitions as I'll be using the matrix method to show that various connexive relevant logics fail to validate target formulas (usually those of the form $\neg(A \to B)$) by constructing matrix models of $\mathbf{L}$ with homomorphisms which render the target formula undesignated. More details on this method, and extensions thereof, can be found in any textbook on algebraic logic, such as [15]. The methods I use here are a simplified version of the usual methods in abstract algebraic logic, as I deal here with logics as FMLA systems, in the terminology of [17].

Note that all the results I obtain using matrices will take the form of providing counterexamples to formulas, i.e., showing that some formula A is not a theorem of the target logic $\mathbf{L}$. To do this, it suffices to find some matrix model (over an algebra in the appropriate language) which models all the theorems of the logic while not modeling the target, counterexampled, formula. This does not require proving or assuming the completeness of the logic with respect to any class of matrices, but only requires that the logic be sound with respect to the matrix and interpretation given. Completeness results of the usual sort for matrix models are available, using the usual relationship between *Tarskian consequence relations* and classes of matrix models (for details, see [15]), since logics understood as FMLA systems are a special case of logics understood as SET-FMLA systems (see [17, p. 199 f.] for discussion), but will not be needed here, so I omit them.[7]

2.2. **Axiomatisations of Some Relevant Logics.** The logic $\mathbf{B}$, in the language with connectives $\neg, \to, \wedge, \vee$, can be presented in terms of the following axioms:

(Id) $A \to A$

($\wedge$E) $(A_1 \wedge A_2) \to A_i$ $\quad i \in \{1, 2\}$

($\wedge$I) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

($\vee$I) $A_i \to (A_1 \vee A_2)$ $\quad i \in \{1, 2\}$

($\vee$E) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

(Dist) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

[7] If you want to have a Tarskian consequence relation associated with the logics presented here, one can simply consider the *global* consequence relation associated with the axiom systems, in much the same way that one considers global consequence relations associated to modal logics (for the reason that in these systems, some of the rules are "rules of proof" in the sense of [34] – in particular, this holds of (rAdj) for reasons discussed in [4]. For example, consider the treatment in [8, p. 35 f.].

(DNi) $A \to \neg\neg A$
(DNe) $\neg\neg A \to A$

and rules:

$$\frac{A \to B \qquad A}{B}\ (\text{rMP}) \qquad\qquad \frac{A \to B}{\neg B \to \neg A}\ (\text{rCont})$$

$$\frac{A \qquad B}{A \wedge B}\ (\text{rAdj})$$

$$\frac{A \to B}{(C \to A) \to (C \to B)}\ (\text{rB}) \qquad\qquad \frac{A \to B}{(B \to C) \to (A \to C)}\ (\text{rB}')$$

When discussing axioms for logics, I'll combine both (DNi) and (DNe) into one axiom, (DNE), so that what it means to say that a logic **L** includes this is that

$$\{p \to \neg\neg p, \neg\neg p \to p\} \subseteq \mathbf{L}$$

Definition 2.9 (Derivations)**.** A *derivation* in an axiomatically presented logic **L** is a *tree* each node of which is labeled by a formula, such that the leaves are labeled by (instances of) axioms and each non-leaf node is labeled by a formula which results by an application of one of the rules from formulas labeling the immediate successor nodes.[8]

When A labels the root node of a proof tree of **L**, I'll write $\vdash_{\mathbf{L}} A$.[9]

Note that in any logic which is closed under the rules (rB) and (rB′)[10], we can show the logic to be closed under the further rule:

$$\frac{A \to B \qquad B \to C}{A \to B}\ (\text{rHS})$$

Among the extensions of **B**, those which I'll have cause to mention here are those which add the following axioms:

(Cont) $(A \to B) \to (\neg B \to \neg A)$
(CM) $(\neg A \to A) \to A$
(B) $(A \to B) \to ((C \to A) \to (C \to B))$
(B′) $(A \to B) \to ((B \to C) \to (A \to C))$
(W) $(A \to (A \to B)) \to (A \to B)$

[8]Where space makes presenting the tree impracticable, I'll write these trees as linear sequences of formulas in a standard way.

[9]I'll use similar notation when describing membership in logics more abstractly described, i.e. writing $\vdash_{\mathbf{L}} A$ in place of $A \in \mathbf{L}$.

[10]Alternate names for these principles are "rule prefixing" and "rule suffixing", respectively. I use the letter-names for shortness, but will sometimes refer to the pair of these as "affixing" rules.

(C) $(A \to (B \to C)) \to (B \to (A \to C))$
(K) $A \to (B \to A)$

and the rule:

$$\frac{A}{(A \to B) \to B}\ (\mathsf{rCI})$$

The extensions of **B** which will come up include the following:

E: $\mathbf{B} \oplus \{$(Cont), (CM), (B), (B′), (W)$\}$ (closed under (rCI)).
RW$^-$: $\mathbf{B} \oplus \{(\mathsf{B}), (\mathsf{B}'), (\mathsf{C})\}$.
RW: $\mathbf{RW}^- \oplus \{$(Cont)$\}$
R: $\mathbf{RW} \oplus \{(\mathsf{W})\}$
J$^+$: $\mathbf{R}^+ \oplus \{(\mathsf{K})\}$.

Note that $\mathbf{J}^+$ is the positive fragment of intuitionistic logic (it's included here just for comparison purposes).

Given one of these logics, or any other logic, I'll specify their *connexive extensions* according to the following formula:

LA: $\mathbf{L} \oplus \{$(Ari1), (Ari2)$\}$
LB: $\mathbf{L} \oplus \{$(Boe1), (Boe2)$\}$

where we name the (positive) connexive theses:

(Ari1) $\neg(A \to \neg A)$
(Ari2) $\neg(\neg A \to A)$
(Boe1) $(A \to B) \to \neg(A \to \neg B)$
(Boe2) $(A \to \neg B) \to \neg(A \to B)$

This is enough to be going on with for now. During the proceedings (particularly in § 4.2) I'll introduce some *ad hoc* logics via axiomatisations, but will leave those evils to the section thereof.

3. Unsavouriness and Triviality

3.1. **Unsavouriness.** Let's now recapitulate the proof in [33, Thm. 3.17(2)] that almost all connexive relevant logics prove $\neg(p \to q)$, and hence every formula $\neg(A \to B)$. For sake of comparison, I'll break these up into two parts.

Proposition 3.1. Let **L** validate the axioms ($\wedge$E) and (Ari) and be closed under the rules (rMP), (rCont), (rB), and (rB′). Then $\vdash_{\mathbf{L}} \neg(p \to \neg q)$.

Proof. The derivation is as follows:

$$
\dfrac{
\dfrac{\dfrac{(p \wedge q) \to q}{\neg q \to \neg(p \wedge q)}\,\text{(rCont)}}{(p \to \neg q) \to (p \to \neg(p \wedge q))}\,\text{(rB)}
\qquad
\dfrac{(p \wedge q) \to p}{(p \to \neg(p \wedge q)) \to ((p \wedge q) \to \neg(p \wedge q))}\,\text{(rB}'\text{)}
}{
\dfrac{\dfrac{(p \to \neg q) \to ((p \wedge q) \to \neg(p \wedge q))}{\neg((p \wedge q) \to \neg(p \wedge q)) \to \neg(p \to \neg q)}\,\text{(rCont)}}{\neg(p \to \neg q)}\,\text{by (Ari), (rMP)}
}\,\text{(rHS)}
$$

Note that (Ari) is only needed in the last step, so that all others are derivable in any logic with the other principles in question. ☐

Note that this first 'unsavoury' principle is characteristic of what Estrada-González and Cano-Jorge [12, p. 192] call "ultra-Abelardian" logics. They suggest that Abelard gave reasons to reject any implication of the form $A \to \neg B$, and that might provide a reason not to be too upset with the unsavouriness involved in Prop. 3.1. There is a question, then, of whether it's worth avoiding this first principle, though it's not a question I'll go into here. For now I'll just move on to the second part using it and some additional principles.

Proposition 3.2. Suppose $\mathbf{L}$ satisfies the constraints of Prop. 3.1 and furthermore $\mathbf{L} \cup \{\neg\neg p \to p, p \to \neg\neg p\} \neq \varnothing$. Then $\vdash_{\mathbf{L}} \neg(p \to q)$.

Proof. First suppose that $\vdash_{\mathbf{L}} q \to \neg\neg q$, then we can reason as follows:

$$
\dfrac{
\dfrac{\dfrac{q \to \neg\neg q}{(p \to q) \to (p \to \neg\neg q)}\,\text{(rB)}}{\neg(p \to \neg\neg q) \to \neg(p \to q)}\,\text{(rCont)}
\qquad
\dfrac{\Delta}{\neg(p \to \neg\neg q)}
}{\neg(p \to q)}\,\text{(rMP)}
$$

Where Δ is the derivation in Prop. 3.1, but with $\neg q$ uniformly substituted for q. The proof in the case that $\vdash_{\mathbf{L}} \neg\neg p \to p$ is similar, just using (rB′) rather than (rB). ☐

These have some immediate corollaries for the inconsistency of such logics:

Corollary 3.3. If $\mathbf{L}$ satisfies the conditions of Prop. 3.1, and is closed under substitutions, then whenever $\vdash_{\mathbf{L}} A \to \neg B$, it follows that $\{A \to \neg B, \neg(A \to \neg B)\} \subseteq \mathbf{L}$.

Corollary 3.4. If $\mathbf{L}$ satisfies the conditions of Prop. 3.1, Prop. 3.2, and is closed under substitutions, then whenever $\vdash_{\mathbf{L}} A \to B$, it follows that $\{A \to B, \neg(A \to B)\} \subseteq \mathbf{L}$.

So connexive relevant logics, at least in the usual sense, are not just contradictory as logics, but they are *really rather* contradictory, as most satisfy the constraints of both Props. 3.1 and 3.2.[11] Of particular note in the case of generating contradictions there is the well-known recipe making use of the fact that basically all seriously proposed relevant logics (in the full language) prove the following formula, directly contradicting (Ari):

$$(p \wedge \neg p) \to \neg(p \wedge \neg p)$$

which follows directly from ($\wedge$E) and (rCont).

3.2. **Triviality.** As is now well known the full logic **R** trivialises its connexive extensions. This is for the simple reason that it trivialises *all* of its contradictory extensions, as was shown by Maksimova [21] and Meyer (reported in [2, §29.11]).[12] Recently [28, Thm. 2] it has been shown that $\mathbf{R}_{\to,\neg}$ also trivialises its connexive extensions, in particular that $\mathbf{RA}_{\to,\neg}$ is trivial. Just as in the case of full **R**, though, this can be seen as a consequence of the fact that $\mathbf{R}_{\to,\neg}$ trivialises all of its contradictory extensions.

Theorem 3.5. Let **L** be a logic s.t.

(1) $\mathbf{R}_{\to,\neg} \subseteq \mathbf{L}$
(2) for some $A \in Fm$, $\{A, \neg A\} \subseteq \mathbf{L}$

Then $\mathbf{L} = Fm$.

Proof. First note that since we have the following, by applications of (C) and (Cont):

$$\vdash_{\mathbf{R}_{\to,\neg}} A \to (\neg A \to \neg(A \to A))$$

it follows from (2) that:

$$\vdash_{\mathbf{L}} \neg(A \to A)$$

Let σ_p be the constant p substitution, for p some atomic formula. But then $\vdash_{\mathbf{L}} \neg(\sigma_p A \to \sigma_p A)$, as $\sigma_p(\neg(A \to A)) = \neg(\sigma_p A \to \sigma_p A)$ and **L** is a logic. Furthermore (as follows from the results in [1]):

$$\vdash_{\mathbf{R}_{\to,\neg}} (p \to p) \to (\sigma_p A \to \sigma_p A)$$

from which it follows, by (1), (rCont), and (rMP), that:

$$\vdash_{\mathbf{L}} \neg(p \to p).$$

[11] In particular, most commonly studied relevant logics are expansions or extensions of the basic system **B** (this is the case of the systems mentioned in [33, ch. 4], for example). Indeed, the constraints, besides (Ari), needed for unsavouriness are also satisfied in the system **BB** introduced in [19]. So the "relevant" part of almost all connexive relevant systems will satisfy all the parts of the results needed besides (Ari), and the "connexive" part will supply at least (Ari). Of course, one could focus on other systems, such as **CB** of [32], but I'll leave that aside here, except to note that it's an intensional system in line with the kind of systems discussed in § 4.3.

[12] To my knowledge, these results were obtained independently.

Hence taking $\sigma(p) = q \to (q \to q)$:

$$\vdash_{\mathbf{L}} \neg((q \to (q \to q)) \to (q \to (q \to q)))$$

But now note the following:

$$\vdash_{\mathbf{R}_{\to,\neg}} q \to ((q \to (q \to q)) \to (q \to (q \to q)))$$

Here's the proof:

(1)	$(q \to (q \to q)) \to ((q \to q) \to (q \to (q \to q)))$	(B)
(2)	$((q \to q) \to (q \to (q \to q))) \to ((q \to (q \to q)) \to (q \to (q \to (q \to q))))$	(B)
(3)	$(q \to (q \to q)) \to ((q \to (q \to q)) \to (q \to (q \to (q \to q))))$	from (1), (2)
(4)	$(q \to (q \to q)) \to (q \to (q \to (q \to q)))$	from (3) by (W)
(5)	$q \to ((q \to (q \to q)) \to (q \to (q \to q)))$	from (4) by (C)

In fact, all the steps up to, but not including, (5) are provable in relevant logics significantly weaker than $\mathbf{R}_{\to}$.[13]

Thus we have, by (Cont), that:

$$\vdash_{\mathbf{L}} \neg q$$

and so, substituting $\neg q$ for q, and by (DNE) we have that $\vdash_{\mathbf{L}} q$, and so for any formula B, $\vdash_{\mathbf{L}} B$. Hence $Fm \subseteq \mathbf{L}$, and so $\mathbf{L}$ is trivial. □

Corollary 3.6. $\mathbf{RA}_{\to,\neg}$ is trivial.

Proof. By Thm. 3.5, it suffices to show that $\mathbf{RA}_{\to,\neg}$ is contradictory. In [28, Lem. 1], it is shown that for any p, q, $\vdash_{\mathbf{RA}_{\to,\neg}} (p \to \neg p) \to q$. So fixing the substitution $\sigma(p) = A$ and $\sigma(q) = \neg(A \to \neg A)$, we obtain:

$$\vdash_{\mathbf{RA}_{\to,\neg}} (A \to \neg A) \to \neg(A \to \neg A)$$

but because of the presence of (Ari), we have that:

$$\vdash_{\mathbf{RA}_{\to,\neg}} \neg((A \to \neg A) \to \neg(A \to \neg A))$$

and the result follows. □

So $\mathbf{R}_{\to,\neg}$ is a bad place to store one's connexive theses. Indeed, the same holds for the contraction-free subsystem $\mathbf{RW}_{\to,\neg}$.

Remark 3.7. Weiss [41, note 31] notes that if we consider the intensional fragment of **RW**, with (Cont) instead of just (rCont), then even the addition of (Ari) causes $\to$ to be symmetric. The key lemma here

$$\vdash_{\mathbf{RW}_{\to,\neg}} \neg((A \to B) \to \neg(A \to B)) \to ((B \to A) \to (A \to B))$$

was proved by Marko Malink, as communicated to me by Weiss, and is straightforward to prove in a sequent system, such as that discussed in [7, § 9.1.2] *minus* the contraction rule as follows (I'll name rules following the

[13]Note however that for the full result we need both (W) and (C), which has the result of pinning down **R** among the usual relevant logics – at least, those of the 'Anderson-Belnap' school.

kind of naming convention employed in [7], and write labels on the left to distinguish from the axiomatic derivations presented elsewhere):

$$
\begin{prooftree}
\AxiomC{$A \succ A$}
\AxiomC{$B \succ B$}
\AxiomC{$A \succ A$}
\AxiomC{$B \succ B$}
\LeftLabel{$(\to \succ)$}
\BinaryInfC{$A \to B, A \succ B$}
\LeftLabel{$(\to \succ)$}
\BinaryInfC{$A \to B, B \to A, B \succ B$}
\LeftLabel{$(\to \succ)$}
\BinaryInfC{$A \to B, B \to A, A \to B, A \succ B$}
\LeftLabel{$(\succ \neg)$}
\UnaryInfC{$B \to A, A \to B, A \succ B, \neg(A \to B)$}
\LeftLabel{$(\succ \to)$}
\UnaryInfC{$B \to A, A \succ B, (A \to B) \to \neg(A \to B)$}
\LeftLabel{$(\neg \succ)$}
\UnaryInfC{$\neg((A \to B) \to \neg(A \to B)), B \to A, A \succ B$}
\LeftLabel{$(\succ \to)$}
\UnaryInfC{$\neg((A \to B) \to \neg(A \to B)), B \to A \succ A \to B$}
\LeftLabel{$(\succ \to)$}
\UnaryInfC{$\neg((A \to B) \to \neg(A \to B)) \succ (B \to A) \to (A \to B)$}
\end{prooftree}
$$

Note that the proof system here is essentially Gentzen's **LK** but without the weakening and contraction rules, which is adequate for $\mathbf{RW}_{\to,\neg}$.

Note: since first writing this remark, Shawn Standefer has communicated to me a simple proof showing that unsavoury extensions of $\mathbf{RW}_{\to,\neg}$ are trivial:

(1)	$\neg p \to ((\neg p \to q) \to q)$	(CI)
(2)	$\neg((\neg p \to q) \to q) \to p$	1 (rCont), (DNE)
(3)	$\neg((\neg p \to q) \to q)$	Unsavouriness
(4)	p	2,3 (rMP)

This just goes to show that **RW** is also a bad place to store your connexive theses.

Furthermore, it is also rather strong, and so these negative results are preserved to a number of candidate intensional connexive relevant logics.

If, however, we go a bit weaker, in the direction of **E**, we can do better.

3.3. **EA and its Relevance Properties.** We know that $\vdash_{\mathbf{EA}} \neg(p \to q)$ holds, but the interesting upshot of [33, Cor. 3.10(1)] is that this fact does not prevent **EA** from having desirable relevance properties. In particular, it satisfies Belnaps VARIABLE SHARING criterion:

Proposition 3.8. [33, Cor. 3.10(1)] If $\vdash_{\mathbf{EA}} A \to B$ then A and B have an atomic subformula in common.

as well as the ACKERMANN property, which Anderson and Belnap [2] touted as the further property justifying **E** as against **R**:

Proposition 3.9. [33, Cor. 3.11(2)]
If $\vdash_{\mathbf{EA}} A \to (B \to C)$ then A has at least one occurrence of $\to$.

So the fact that **EA** is unsavoury does not get in the way of its being relevant in a rather strong sense.[14] Furthermore, these facts even allow one to construct *consistent* theories on the basis of this contradictory logic (as discussed in [22]).

Nonetheless, it seems desirable to avoid unsavouriness. Even if it doesn't trivialise the logics in question, or undermine their claims to relevance, it's an odd theory of implication that says "no!" to every implication statement (even if it also says "yes!" to some of them). To that end, let's turn our attention to our real focus, how to avoid unsavouriness.

4. Avoiding Unsavouriness

Looking back at the proof of unsavouriness, there are three candidate principles we could drop (excepting, of course, the connexive theses). These are:[15]

(1) *Contraposition* in its (rCont) or its (Cont) guise.
(2) Lattice operators – especially ($\wedge$E) and ($\vee$I).
(3) The *affixing* rules (rB) and (rB$'$).

If we drop the first, we obtain an *ill-contraposing* logic; if we drop the second, we obtain an *intensional* logic; if we drop the third, we obtain an *ill-affixing* logic. Let's consider the options for these three avenues in turn.

4.1. **Ill-Contraposing Logics.**

The first, and most popular option among connexivists, is to reject contraposition in the rule form:

$$\text{(rCont)}\ \frac{A \to B}{\neg B \to \neg A}$$

In [38], a connexive logic **C** is obtained by adding to $\mathbf{J}^+$ a negation connective $\sim$ (so-called "strong" negation, in contrast to the "weak" negation usually defined in intuitionistic logic) obeying the following biconditional axioms:

- $\sim\sim A \leftrightarrow A$
- $\sim (A \wedge B) \leftrightarrow (\sim A \vee \sim B)$
- $\sim (A \vee B) \leftrightarrow (\sim A \wedge \sim B)$
- $\sim (A \to B) \leftrightarrow (A \to \sim B)$

[14] Sylvan [33, §3] used these facts about **EA** to argue, against Anderson and Belnap, that **E** did not, contrary to appearances, constitute the strongest logic satisfying the Variable Sharing and Ackermann properties.

[15] It was pointed out to me, by Luis Estrada-González, that another option is to reject closure under substitution. I will forgo considering that option here, but note that it is another potential avenue.

The first three of these hold in all the relevant logics under consideration here, swapping $\sim$ for $\neg$: indeed, they are, in a sense, definitive of $\neg$'s being the *De Morgan negation* of **FDE**, which is a subsystem of all the usual relevant logics. It is the final biconditional, which combines (Boe1) with its converse, which does the trick, allowing for all the connexive theses to be provable, while retaining non-triviality.

Note, however, that if we were to add (rCont) to **C**, we'd be able to mimic the kind of reasoning done in the proof of Thm. 3.5, and obtain a trivial system.[16] So trading out (rCont) for these different axioms for negation is a key part of the move. Doing so does, however, allow us to avoid proving every negated implication (we'll come back to this point in § 4.2.4).

Given this, then, a natural strategy to pursue in giving ill-contraposing connexive relevant logics is to mimic this trick, but trading out the positive basis $\mathbf{J}^+$ for that of some relevant logic, such as $\mathbf{R}^+$. Precisely this last move has been done recently, and the results turn out to be fairly natural. We get natural deduction systems [16] and natural frame-theoretic semantics [26, 40]. For the latter bit, the trick is to replace the usual frame-theoretic treatment for negation among relevant logics (the "star" operation of [31], and its interaction with the relation interpreting implication) with a *four-valued* semantic machinery.[17] Related work has been done by Estrada-González and Tanús-Pimentel [13], who start from a non-contraposing variant of a logic near **B**, and consider VARIABLE SHARING-like properties which obtain there.[18] In such a setting, there is a straightforward way to characterise this kind of 'strong' negation, and gently lay it on top of whatever positive logic you like. So one could just choose a relevant logic to lay the connexive theses upon.

Of the three approaches this is the one I have the least to say about. First of all, it's been done. Second, though, is that for reasons I mentioned in the introduction, dropping contraposition involves a fairly significant departure from the theory of meaning which seems to be the best fit for relevant logics – namely, the frame/algebraic semantics of gaggle theory. So while this is an option, it's not the one I'll focus on here any more than I've done so already.

4.2. **Ill-Affixing Logics.** The final remaining option I'll consider here concerns dropping one of the affixing rules:

[16] As $\{(p\wedge \sim p) \to \sim(p\wedge \sim p), \sim((p\wedge \sim p) \to \sim(p\wedge \sim p))\} \subseteq \mathbf{C}$.

[17] The difference between the four-valued approach to negation, and the star-based one, is the crux of the dispute in [9, 5].

[18] That is, they investigate properties like VARIABLE SHARING which hold for connexive logics – they do not investigate connexive logics which satisfy VARIABLE SHARING.

$$\dfrac{A \to B}{(C \to A) \to (C \to B)}\ \text{(rB)} \qquad \dfrac{A \to B}{(B \to C) \to (A \to C)}\ \text{(rB}'\text{)}$$

Dropping at least one of these allows us to retain the full vocabulary, including the lattice connectives, as well as contraposition in various forms. It does, however, have the somewhat undesirable consequence of altering the behaviour of $\to$ to no longer have the usual tonicity type. That is to say, it is in lieu of these two rules that we have, respective, that $\to$ is *monotone* in the second argument place, and *antitone* in the first. That is, in algebraic semantics corresponding to these logics, for instance those given in terms of *gaggles* [6], which are partially ordered algebras with operations standing in certain relations, we have certain quasi-inequations provable. In particular, these are:

$$\dfrac{a \leq a'}{b \to a \leq b \to a'} \qquad \dfrac{a' \leq a}{a \to b \leq a' \to b}$$

So one clear downside of this strategy is that it cuts against the logical behaviour of $\to$ in much the same way that rejecting contraposition cuts against that of $\neg$.

Having said this, however, nonetheless we can obtain some suprisingly powerful logics by dropping one (or both) of these rules, and adding in other principles. What we can add does seem to rely, to a significant extent, on which of these we remove. We can see right away that keeping (rB$'$) is more difficult that keeping (rB).

Proposition 4.1. If $\mathbf{L}$ includes the axioms (Cont), ($\vee$E), and (DNE), along with the rules (rMP), (rHS), and (rB$'$), then $\vdash_{\mathbf{L}} \neg(\neg A \to B)$.

Proof. Take the following derivation, dual, in a sense, to that from Prop. 3.1:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to (A \vee B)}{\neg(A \vee B) \to \neg A}\ \text{(rCont)}}{(\neg A \to B) \to (\neg(A \vee B) \to B)}\ \text{(rB}'\text{)}}{(\neg A \to B) \to (\neg B \to (A \vee B))}\ (\star) \qquad \dfrac{\dfrac{B \to (A \vee B)}{\neg(A \vee B) \to \neg B}\ \text{(rCont)}}{(\neg B \to A \vee B) \to (\neg(A \vee B) \to (A \vee B))}\ \text{(rB}'\text{)}}{(\neg A \to B) \to (\neg(A \vee B) \to (A \vee B))}\ \text{(rHS)}}{\neg(\neg(A \vee B) \to (A \vee B)) \to \neg(\neg A \to B)}\ \text{(rCont)}}{\neg(\neg A \to B)}\ \text{(Ari), (rMP)}$$

Note that the derivation step labeled ($\star$) can be derived by appeal to (rMP) as well as the following theorem, using an instance of (Cont) as the left-most premise:

$$\dfrac{(\neg(A \vee B) \to B) \to (\neg B \to \neg\neg(A \vee B)) \qquad \dfrac{(A \vee B) \to \neg\neg(A \vee B)}{(\neg B \to \neg\neg(A \vee B)) \to (\neg B \to (A \vee B))}\ \text{(rB}'\text{)}}{(\neg(A \vee B) \to B) \to (\neg B \to (A \vee B))}\ \text{(rHS)}$$

$\square$

Note that if we have (DNE), we can easily trade this result in for one showing that $\vdash_{\mathbf{L}} \neg(A \to B)$, as before. This goes to show that we are somewhat limited by the inclusion of (rB′), but below I'll consider three *ad hoc* logics which show the possibilities if we reject this.

4.2.1. ***Thing***$_1$. Our first *ad hoc* logic, **Thing**$_1$, drops both of (rB) and (rB′), while retaining (Cont) and both (Ari) and (Boe). Hence it is a truly connexive relevant logic (in axiomatising these, I omit (rMP) since it is built into the definition of something's being a logic).

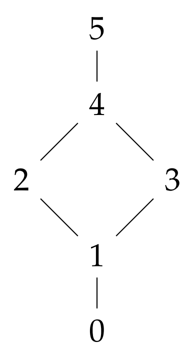
(Id)	$A \to A$	
(W)	$(A \to (A \to B)) \to (A \to B)$	
(∧E)	$(A_1 \wedge A_2) \to A_i$	$i \in \{1,2\}$
(∧I)	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$	
(∨E)	$A_i \to (A_1 \vee A_2)$	$i \in \{1,2\}$
(∨E)	$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$	
(Dist)	$(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$	
(WB)	$((A \to B) \wedge (B \to C)) \to (A \to C)$	
(Cont)	$(A \to \neg B) \to (B \to \neg A)$	
(DNE)	$\neg\neg A \to A$	
(CM)	$(A \to \neg A) \to \neg A$	
(LEM)	$A \vee \neg A$	
(Ari)	$\neg(A \to \neg A), \neg(\neg A \to A)$	
(Boe)	$(A \to B) \to \neg(A \to \neg B), (A \to \neg B) \to \neg(A \to B)$	

$$\frac{A \qquad B}{A \wedge B}\,(\mathrm{rAdj})$$

In the argument below, in order to show that the logics are savoury, we'll show that we can invalidate the special case $\neg(A \to A)$.

Proposition 4.2. $\nvdash_{\mathbf{Thing}_1} \neg(A \to A)$

Proof. The following countermodel suffices – we start with the hasse diagram:



where we fix $\neg x = 5 - x$ and the definition of $\to$ to be:

$\to$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	0	1	1	1	1	1
2	0	0	5	0	1	1
3	0	0	0	5	1	1
4	0	0	0	0	1	1
5	0	0	0	0	0	1

and take the designated values to be $\{x \mid 1 \leq x\}$. Note that $\neg(2 \to 2) = 0$. □

It's worth noting that **Thing**$_1$ is a sublogic of the system **M3V**, which is the complete logic of the matrix used in [33, §3.6] to show that **EA** satisfies ACKERMANN. This system was independently studied in [12]. As a result, **Thing**$_1$ also satisfies ACKERMANN. I don't as of yet know, however, whether **Thing**$_1$ satisfies VARIABLE SHARING (and the argument to this effect in [33] relies on the fact that **EA** is an extension of **E** by only negated implication theses, which is not the case when we also add (Boe)).[19]

Having said this, note that there is at least one *rather* irrelevant extension of **Thing**$_1$ to which the given countermodel applies, namely the expansion by the further axioms:

(Safety) $(A \wedge \neg A) \to (B \vee \neg B)$
(3-isch) $A \vee (A \to B)$

But these principles are deeply undesirable in a relevant setting.[20] In fact, both of the following logics, and countermodels, I'll give permit the inclusion of these principles, though I'll not mention this fact any more than I just have.

4.2.2. ***Thing***$_2$. In the following system, **Thing**$_2$, we retain (rB) and (Cont), but drop (Boe), obtaining a merely connexive-isch relevant logic.

(Id) $A \to A$
(W) $(A \to (A \to B)) \to (A \to B)$
($\wedge$E) $(A_1 \wedge A_2) \to A_i$ $\quad i \in \{1,2\}$
($\wedge$I) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
($\vee$E) $A_i \to (A_1 \vee A_2)$ $\quad i \in \{1,2\}$
($\vee$E) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

[19]Note that, as a consequence of [33, Thm. 3.16], $\nvdash_{\mathbf{EA}} (A \to B) \to \neg(A \to \neg B)$, and so a distinct proof of VARIABLE SHARING is needed, if it holds at all, for **EAB**=**EB**.

[20]An argument against the former can be found in [36].

(Dist) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
(WB) $((A \to B) \wedge (B \to C)) \to (A \to C)$
(Cont) $(A \to \neg B) \to (B \to \neg A)$
(DNE) $\neg\neg A \to A$
(CM) $(A \to \neg A) \to \neg A$
(LEM) $A \vee \neg A$
(Ari) $\neg(A \to \neg A), \neg(\neg A \to A)$

$$\frac{A \qquad B}{A \wedge B}\,(\text{rAdj}) \qquad\qquad \frac{A \to B}{(C \to A) \to (C \to B)}\,(\text{rB})$$

Proposition 4.3. $\nvdash_{\mathbf{Thing_2}} \neg(A \to A)$.

Proof. We can obtain a countermodel by taking the three-valued matrix for **LP** (see [30] for its most famous introduction), with 2 as just true, 1 as both true and false, and 0 as just false[21], along with implication defined in terms of the following table:

$\to$	0	1	2
0	2	1	1
1	0	1	1
2	0	0	2

Fixing the designated values to be $\{1, 2\}$, we have $\neg(0 \to 0) = 0$. □

The major downside here is the loss of (Boe), but it still serves as some proof of concept that we can retain (rB) and axiom contraposition while avoiding unsavouriness. Furthermore, it is clear that $\mathbf{Thing_2}$ is a subsystem of **EA**, and thus also satisfies VARIABLE SHARING and ACKERMANN.

Now let's see an example where we go for only the rule form of contraposition.

4.2.3. ***Thing***$_3$. It turns out that if we trade out (Cont) for (rCont), we can have (rB) and (Boe) as well.

(Id) $A \to A$
(W) $(A \to (A \to B)) \to (A \to B)$
($\wedge$E) $(A_1 \wedge A_2) \to A_i$ $\quad i \in \{1, 2\}$
($\wedge$I) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
($\vee$I) $A_i \to (A_1 \vee A_2)$ $\quad i \in \{1, 2\}$
($\vee$E) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
(Dist) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
(WI) $((A \to B) \wedge A) \to B$

[21]This is the naming convention used in [25].

(WB) $((A \to B) \wedge (B \to C)) \to (A \to C)$
(DNE) $\neg\neg A \to A$
(CM) $(A \to \neg A) \to \neg A$
(LEM) $A \vee \neg A$
(Ari) $\neg(A \to \neg A), \neg(\neg A \to A)$
(Boe) $(A \to B) \to \neg(A \to \neg B), (A \to \neg B) \to \neg(A \to B)$

$$\frac{A \quad B}{A \wedge B}\ (\text{rAdj}) \qquad \frac{A \to B}{(C \to A) \to (C \to B)}\ (\text{rB}) \qquad \frac{A \to \neg B}{B \to \neg A}\ (\text{rCont})$$

Proposition 4.4. $\nvdash_{\mathbf{Thing}_3} \neg(A \to B)$.

Proof. We can use the matrix described in Prop. 4.3, trading out the definition of $\to$ for the following:

$\to$	0	1	2
0	1	1	1
1	0	1	2
2	0	0	1

Then with $\{1, 2\}$ as the designated values once again, we have $\neg(1 \to 2) = \neg 2 = 0$. $\square$

While rather *ad hoc*, these three systems at least provide proof of concept, that we can obtain the desired results while dropping (rB′) (perhaps in addition to (rB)), and still build in a number of axioms and principles.

4.2.4. *A Less Ad Hoc Ill-Affixing Logic.* A more complex, but less *ad hoc*, example of a system can be obtained by considering the 'strong' implication, in the style of [35], defined in the connexive system **C3** in [27]. The primitive connexive implication of **C**, and that of the expansion **C3** of **C**, doesn't obey contraposition, but in [27] the authors add a defined connective which validates the axiom form. I'll alter their notation a bit, using $\to$ for the primitive implication, $\neg$ for negation, and $\prec$ for the defined implication[22], so that:

$$A \prec B := (A \to B) \wedge (\neg B \to \neg A)$$

As they discuss, this system proves (Cont), ($\wedge$E) and ($\vee$I), (DNE), and (Ari), and is closed under (rHS), while avoiding both forms of unsavouriness, and falsifying $\neg(\neg A \prec B)$.[23] Thus it is not closed under (rB′) as, given Prop. 4.1, if it were it would prove $\neg(\neg A \prec B)$.[24]

[22] I reserve "$\Rightarrow$" for the metalanguage implication.

[23] In [27] a countermodel is given to show that **C3** does not prove $\neg(A \prec B)$, and that model can easily be adapted to show that it also does not prove $\neg(A \prec \neg B)$ or $\neg(\neg A \prec B)$.

[24] The task of finding a concrete counterinstance of (rB′) in $\mathbf{C3}_{\prec}$ is left to the interested reader.

C3 with $\prec$ as the main implication connective provides an "in the wild" example of a connexive logic which avoids at least some irrelevant theses (such as $A \prec (B \prec A)$, which is not provable), and which avoids unsavouriness by means of going ill-affixing.

4.2.5. *Issues with Extensions and Meanings.* It is important to note that (rB) and (rB′) play a similar role to the meaning of $\to$, in the broadly gaggle-theoretic framework, that (rCont) does to that of $\neg$. So the reasons to want to hold on to (rCont) would seem also to push one to want to hold onto (rB) and (rB′). One noteworthy point here is that rejecting (rB′) is a common move in *conditional* logics in the 'Lewis-Stalnaker-Chellas-Segerberg-Kratzer' dynasty.[25] So if we consider not relevant logics as concerning entailment, primarily, but as concerning conditionals, then there is some wiggle room as far as rejecting (rB′) goes. Indeed, connexive and conditional logics do have some overlap, e.g. [39].

Beyond conceptual difficulties, there are also theoretical difficulties. Removing one of (rB), (rB′) puts constraints on what sort of additions we can make to the language – as discussed in [6], these rules are tightly tied to properties of the ternary relation, in the usual ternary relation semantics for relevant logics, and removing these will rule out giving such a semantics without some tweaks. Such tweaks are available, for instance in the "neighbourhood ternary relation" framework developed by Lavers [19] and since then also by Ferenz and Tedder [14].

4.3. **Intensional Logics.** As we saw, $\mathbf{R}_{\to,\neg}$ is inhospitable to connexive theses. We can, however, cook up weaker intensional relevant systems which provide a much more inviting home. In fact, by going only slight weaker, to $\mathbf{RW}_{\to,\neg}$, and systems in its vicinity, we have options.

In [29], a system in this vicinity is obtained which allows for connexive principles, and a three-valued characteristic matrix semantics, while avoiding unsavouriness. The trick relies on the use of an intuitionistic-style negation, and so involves the rejection of (DNe). Of course, having a three-valued semantics makes this difficult for VARIABLE SHARING, and indeed their system proves $(A \to A) \to (B \to B)$ for any A, B. Nonetheless, their way of proceeding does show that an intensional-fragment-extension strategy can work. We just need to find a way to do so while retaining relevance.

I'll investigate a related nearby option below, where we have **FDE** as a subsystem. Namely, this system is $\mathbf{RW}^-$, retaining (DNE) but removing (Cont), in favour of (rCont).

[25]For various references here, one can see the SEP article [11].

But before digging into that issue, let's consider a further issue, namely, the potential for adding *intensional* conjunction and disjunction to intensional logics.

4.3.1. *Adding Fusion and Fission.* In the setting of $\mathbf{R}_{\to,\neg}$, we can straightforwardly define two additional intensional connectives recapturing some of the desired behaviour of "and" and "or". These are fusion $\circ$ and fission $+$ (or "intensional conjunction/disjunction"), definable as follows:

$$A \circ B = \neg(A \to \neg B)$$
$$A + B = \neg A \to B$$

With these definitions, we then capture some of the desired properties, namely those captured by the following validities (writing $A \leftrightarrow B \in \mathbf{L}$ as shorthand for $\{A \to B, B \to A\} \subseteq \mathbf{L}$):

(1) $$((A \circ B) \to C) \leftrightarrow (A \to (B \to C))$$

(2) $$((A \circ B) \to C) \leftrightarrow (A \to (\neg B + C)$$

(3) $$((A + B) \circ \neg A) \to B$$

(4) $$(A \circ B) \leftrightarrow \neg(\neg A + \neg B)$$

(5) $$(A + B) \leftrightarrow \neg(\neg A \circ \neg B)$$

Item (4) indicates that $\to$ *residuates* $\circ$: to put these in terms of the algebraic behaviours of their associated operators, relative to an order $\leq$, this means that:

$$a \circ b \leq c \iff a \leq b \to c$$

Item (5) recapitulates the usual behaviour of lattice conjunction/disjunction, mediated by Boolean negation, but now in an intensional setting. Item (6) shows that we recapture a form of *intensional disjunctive syllogism*, which was one of the original motivations for introducing $+$, and related operators (for examples of which, see [20, 24]). Items (7) and (8) then indicate that a De Morgan duality obtains between $\circ$ and $+$, so defined.

The main thing distinguishing $\circ$ and $+$ from their lattice counterparts, then, are the failures of implications, for $i \in \{1, 2\}$:

(6) $$(A_1 \circ A_2) \to A_i$$

(7) $$A_i \to (A_1 + A_2)$$

This is, basically, what sets these apart as *intensional* conjunction and disjunction.

The positive properties (4)–(8) for *defined* $\circ$ and $+$, however, do depend on the properties of $\mathbf{R}_{\to,\neg}$ which, as we've seen, is a bad place to store your connexive principles. So if we want to investigate connexive extensions of intensional relevant logics, we need to go to weaker logics, where some of those positive properties may fail. It is possible to retain at least a rule form of (4) if we take $\circ$ as a defined operator:

$$\frac{(A \circ B) \to C}{\overline{A \to (B \to C)}}$$

and obtain thereby conservative extensions of most of the intensional relevant logics. We don't, however, have any guarantee that doing so will result in conservative extensions of connexive extensions of intensional relevant logics, but as a matter of fact it seems that adding $\circ$ *usually* doesn't do too much damage.[26]

When it comes to $+$, however, things are less clear. It's properties are usually characterised relative to $\circ$, as you can see above, and once we no longer take that connective as defined, we have a choice about how we want $+$ to behave. Basically, there are two options:

(1) Preserve intensional disjunctive syllogism: $(A + B) \leftrightarrow (\neg A \to B)$
(2) Preserve De Morgan duality: $(A + B) \leftrightarrow \neg(\neg A \circ \neg B)$

In certain contexts, such as those which have (Cont) among the axioms, trying to preserve both the above will force the logic to include (C). So while either of the above biconditionals amount to definitions, and so automatically conservative, trying to have both of them may result in non-conservative extensions of the target logic.

So if we want an intensional vocabulary containing analogues of the usual bits of logical vocabulary, including disjunction, we have a choice point. I don't have solid reasons for preferring one or the other, but just flag the issue here. In the result below, we can add $+$ to the target logic in accordance with either (but not both) of the above options, and preserve the results.

4.3.2. ***RWB***$^{-}_{\to,\neg,\circ}$*: A Savoury Connexive Logic Constructible from an Intensional Relevant Logic.* The target system is $\mathbf{RWB}^{-}_{\to,\neg}$, in accordance with our naming conventions from § 2.2, noting that in the context of $\mathbf{RW}^{-}$, all instances of ARISTOTLE are provable from instances of BOETHIUS. It satisfies all the positive criteria, but we need to show that it satisfies two negative criteria, which we can do so with one countermodel.

Theorem 4.5. $\nvdash_{\mathbf{RWB}^{-}_{\to,\neg,\circ}} \neg(A \to B)$ and $\nvdash_{\mathbf{RWB}^{-}_{\to,\neg,\circ}} (A \to B) \to (B \to A)$.

Proof. Take the four element matrix $\langle\{0,1,2,3\},\{0\},\neg,\to,\circ\rangle$, defined:

[26] It is included in [29], and we'll have it in the logic below.

$\circ$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\to$	0	1	2	3
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

	$\neg$
0	2
1	3
2	0
3	1

Note that $\neg(0 \to 0) = 2$ and $(1 \to 2) \to (2 \to 1) = 2$. It is tedious, but straightforward, to verify that this matrix satisfies all the theorems of the target logic.

□

This shows that $\mathbf{RWB}^-_{\to,\neg,\circ}$ is not only a connexive logic, but also avoids unsavouriness.

Note that the matrix used in the proof of Thm. 4.5 is just $\langle \mathbb{Z}_4, +_4 \rangle$, with $+_4 = \circ$ and $x \to y = y -_4 x$, though in this group structure $\neg$ is *not* the group inverse. Given this definition, we can take $\to$ to residuate $\circ$, but with respect to the *discrete* order – that is, $x \circ y = z \iff x = y \to z$ holds. That is, in this model, $x \to y = y -_4 x$. Given that we have a discrete order, in accordance with the results in [37], this model will have $\neg$ as a Kapsner complementation operator just in case it has no fixed point, which this does not.[27]

In fact, that we have this model tells us a few things. Since there is only one designated value and no fixed point for negation, it follows that:

Corollary 4.6 (Consistency)**.** Either $\nvdash_{\mathbf{RWB}^-_{\to,\neg,\circ}} A$ or $\nvdash_{\mathbf{RWB}^-_{\to,\neg,\circ}} \neg A$ holds for any $A \in Fm$.

Unfortunately, it's not clear, to me at least, whether this logic satisfies VARIABLE SHARING. Indeed the above model, and others I've found of it (using `MaGIC`), all seem to validate $(A \to A) \to (B \to B)$, which is a problem. Indeed, I've not been able to find a proof of variable sharing for any logic including (rCont) and (Boe).[28] Nor have I found any proof that $\mathbf{RWB}^-_{\to,\neg,\circ}$, at least, fails to satisfy VARIABLE SHARING. As of now, it's unclear to me whether there is a deeper problem here, or just a failure of imagination. So I propose the following:

[27]This gives some insight into why we don't use the group inverse here, as that does have fixed points. Thanks to Shay Logan and Yale Weiss for helpful discussion surrounding this model and the results obtained from it. Weiss pointed out to me that this style of construction works generally for $\langle \mathbb{Z}_n, +_n \rangle$ for any even $n > 2$, but that it seems unlikely to work for any odd n, as we won't guarantee that there aren't cycles in the table for $\neg$, speaking against their Kapsner complementation-ness.

[28]Including those systems mentioned in § 4.2 which include (Boe).

Open Problem 4.7. Does $\mathbf{RWB}^{-}_{\to,\neg,\circ}$ satisfy VARIABLE SHARING?

In any case, the results in this section are somewhat equivocal, but suggestive.

5. CONCLUSION

This paper has investigated options for avoiding the unsavouriness which seems endemic to connexive relevant logics. I've investigated the prospects of these avenues, suggesting systems for the latter two, which are the lesser studied of the three. While perhaps yet more logics are not needed in the literature on either relevant or connexive logics, I've laid out a few, though mainly by way of illustration. An intended upshot of this, besides the core project of avoiding unsavouriness, is to provide some structure, and a classification of some key types of connexive relevant logics. With any luck, there are other systems worth investigating here. Furthermore, there may be some other positive reasons for investigating connexive extensions of relevant logics – mine here have mainly been historical. In any case, I leave these and any further issues for future work.

REFERENCES

[1] Alan R. Anderson and Nuel D. Belnap. “Modalities in Ackermann's “Rigorous Implication””. *Journal of Symbolic Logic* 24(2):107-111 (1959). `doi.org/10.2307/2964754`

[2] Alan R. Anderson and Nuel D. Belnap. *Entailment: The Logic of Relevance and Necessity* Volume 1. Princeton University Press (1975).

[3] Richard Angell. “A Propositional Logic with Subjunctive Conditionals”. *Journal of Symbolic Logic* 27:327–343 (1962). `doi.org/10.2307/2964651`

[4] Guillermo Badia, Libor Běhounek, Petr Cintula, and Andrew Tedder. “Relevant Consequence Relations: An Invitation”. *Review of Symbolic Logic* (Forthcoming).

[5] Francesco Berto and Greg Restall. “Negation on the Australian Plan”. *Journal of Philosophical Logic* 48(6):1119–1144 (2019). `doi.org/10.1007/s10992-019-09510-2`

[6] Katalin Bimbó and J. Michael Dunn. *Generalized Galois Logics: Relational Semantics of Nonclassical Logical Calculi*. CLSI Publications (2008).

[7] Katalin Bimbó. *Proof Theory: Sequent Calculi and Related Formalisms*. CRC Press (2015).

[8] Petr Cintula and Carles Noguera. *Logic and Implication: An Introduction to the General Algebraic Study of Non-Classical Logics*. Springer (2021)

[9] Michael De and Hitoshi Omori. “There is More to Negation than Modality”. *Journal of Philosophical Logic* 47(2):281–299 (2018). `doi.org/10.1007/s10992-017-9427-0`

[10] J. Michael Dunn. “Gaggle Theory: An Abstraction of Galois Connections and Residuation with Applications to Negation, Implication, and Various Logical Operators.” *Lecture Notes in Computer Science* v.478, ed. J. van Eijck, pp.31–51. Springer (1991). `doi.org/10.1007/BFb0018`

[11] Paul Egré and Hans Rott. “The Logic of Conditionals”. *The Stanford Encyclopedia of Philosophy* (Winter 2021 Edition), E. N. Zalta (ed.), (2021). `plato.stanford.edu/entries/logic-conditionals/`

[12] Luis Estrada-González and Fernando Cano-Jorge. "Mortensen Logics". *Electronic Proceedings in Theoretical Computer Science* 358:189–201 (2022). 10th International Conference on Non-Classical Logics. Theory and Applications (NCL 2022), ed. A. Indrzejczak and M. Zawidzki. `arxiv.org/abs/2204.06731`

[13] Luis Estrada-González and Claudia Lucía Tanús-Pimentel. "Variable Sharing in Connexive Logic". *Journal of Philosophical Logic* 50:1377–1388 (2021). `doi.org/10.1007/s10992-021-09602-y`

[14] Nicholas Ferenz and Andrew Tedder. "Neighbourhood Semantics for Modal Relevant Logics". *Journal of Philosophical Logic* 52:145–181 (2023). `doi.org/10.1007/s10992-022-09668-2`

[15] Josep Maria Font. *Abstract Algebraic Logic: An Introductory Textbook*. College Publications (2016).

[16] Nissim Francez. "Relevant Connexive Logic". *Logic and Logical Philosophy* 28:409–425 (2019). `apcz.umk.pl/LLP/article/view/LLP.2019.007`

[17] Lloyd Humberstone. *The Connectives*. MIT Press (2011).

[18] Andreas Kapsner. "Strong Connexivity". *Thought: A Journal of Philosophy* 1(2):141–145 (2012). `doi.org/10.1002/tht3.19`

[19] Peter Lavers. Generating Intensional Logics. M.A. Thesis, University of Adelaide (1985).

[20] Clarence Irving Lewis. "Implication and the Algebra of Logic". *Mind* 21:522–531 (1912). `jstor.org/stable/2249157`

[21] Larisa Maksimova. "On **E**-Theories". *Algebra i Logika* 9(5):530–538 (1970). (originally in Russian, translated to English in 1971).

[22] Franci Mangraviti and Andrew Tedder. "Consistent Theories in Inconsistent Logics". *Journal of Philosophical Logic* 52(4): 1133–1148 (2023). `doi.org/10.1007/s10992-023-09700-z`

[23] Storrs McCall. "Connexive Implication". *Journal of Symbolic Logic* 31:415–433 (1966). `doi.org/10.2307/2270458`

[24] Robert K. Meyer. *Topics in Modal and Many-Valued Logic*. Ph.D. Dissertation, University of Pittsburgh (1966).

[25] Chris Mortensen. "Aristotle's Thesis in Consistent and Inconsistent Logics". *Studia Logica* 43(1-2):107–116 (1984). `doi.org/10.1007/BF00935744`

[26] Hitoshi Omori. "A Simple Connexive Extension of the Basic Relevant Logic **BD**". *IfCoLog Journal of Logics and Their Applications* 3:467–478 (2016). `collegepublications.co.uk/ifcolog/?00007`

[27] Hitoshi Omori and Heinrich Wansing. "An Extension of Connexive Logic **C**". *Advances in Modal Logic, AiML20*, ed. N. Olivetti, R. Verbrugge, S. Negri, G. Sandu, pp. 503–522. College Publications (2020). `aiml.net/volumes/volume13/Omori-Wansing.pdf`

[28] Hitoshi Omori. "A Tension for Mares". *Organon F* (Forthcoming).

[29] Hitoshi Omori and Andreas Kapsner. "Angell and McCall meet Wansing". *Studia Logica* (Forthcoming).

[30] Graham Priest. "The Logic of Paradox". *Journal of Philosophical Logic* 8(1):219–242 (1979).

[31] Richard Routley and Robert K. Meyer. "The Semantics of Entailment - I". *Truth, Syntax and Modality* (ed.) H. Leblanc, pp. 199-243. North-Holland (1973).

[32] Richard Routley. "Semantics for Connexive Logics I". *Studia Logica* 37:393–412 (1978). `doi.org/10.1007/BF02176171`

[33] Richard Routley, Robert K. Meyer, Val Plumwood, and Ross T. Brady. *Relevant Logics and Their Rivals*, Volume 1. Ridgeview Publishing (1982).

[34] T.J. Smiley. "Relative Necessity". *Journal of Symbolic Logic* 28: 113–134 (1963).

[35] Matthew Spinks and Robert Veroff. "Paraconsistent constructive logic with strong negation as a contraction-free relevant logic". *Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science*, J. Czelakowski (ed.), pp. 323–379. Springer (2018). doi.org/10.1007/978-3-319-74772-9_13

[36] Andrew Tedder. "A Note on **R**-Mingle and the Danger of Safety". *Australasian Journal of Logic* 19(1):51–58 (2022). doi.org/10.26686/ajl.v19i1.7449

[37] Andrew Tedder. "Kapsner Complementation: An Algebraic Take on Kapsner Strong Logics". *Studia Logica* 111:321–352 (2023). doi.org/10.1007/s11225-022-10025-2

[38] Heinrich Wansing. "Connexive Modal Logic". *Advances in Modal Logic*, volume 5, pp. 367–383. King's College Publications (2006). philpapers.org/rec/WANCML-3

[39] Heinrich Wansing and Matthias Unterhuber. "Connexive Conditional Logic. Part I". *Logic and Logical Philosophy* 28(3):567–610 (2018). apcz.umk.pl/LLP/article/view/LLP.2018.018

[40] Yale Weiss. "Semantics for Pure Theories of Connexive Implication". *Review of Symbolic Logic* 15(3):591–606 (2022). doi.org/10.1017/S1755020320000374

[41] Yale Weiss. "Did Aristotle Endorse Aristotle's Thesis? A Case Study in Aristotle's Metalogic". *Notre Dame Journal of Formal Logic* 63(4):551–579 (2022). doi.org/10.1215/00294527-2022-0032

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